

Occhino, Filippo

Working Paper

Existence of Equilibrium for Segmented Markets Models with Interest Rate Monetary Policies

Working Paper, No. 2004-11

Provided in Cooperation with:

Department of Economics, Rutgers University

Suggested Citation: Occhino, Filippo (2004) : Existence of Equilibrium for Segmented Markets Models with Interest Rate Monetary Policies, Working Paper, No. 2004-11, Rutgers University, Department of Economics, New Brunswick, NJ

This Version is available at:

<https://hdl.handle.net/10419/23190>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

Existence of Equilibrium for Segmented Markets Models with Interest Rate Monetary Policies

Filippo Occhino*

February 2004

Abstract

Several papers have recently adopted the segmented markets model as a framework for monetary analysis. The characteristic assumption is that some households never participate in financial markets. This paper proves the existence of an equilibrium for segmented markets models where monetary policy is defined in terms of the short-term nominal interest rate. The model allows to consider the important cases where monetary policy affects output, and responds to any source of uncertainty, including output itself. The assumptions

*Address: Department of Economics, Rutgers University, 75 Hamilton Street, New Brunswick NJ 08901. Email: occhino@rutgers.edu. Web: <http://econweb.rutgers.edu/occhino/>. I would like to thank Robert Lucas and Nancy Stokey for their continuous guidance and encouragement. Any errors are my own. I gratefully acknowledge the financial support of the Compagnia di San Paolo di Torino, the University of Chicago, and the Alfred P. Sloan Foundation.

required for existence constrain the maximum value and the variability of the nominal interest rate. The period utility function is logarithmic. The proof is constructive, and shows how the model can be solved numerically. A similar proof can be used in the case that monetary policy is defined in terms of the bond supply.

Keywords: segmented markets, limited participation, interest rate monetary policy, existence.

JEL Classification Number: C60, E52.

1 Introduction

Several papers, among which Alvarez, Lucas and Weber (2001), Lahiri, Singh and Vegh (2003) and Occhino (2004), have recently adopted the segmented markets model as a framework for monetary analysis. The characteristic assumption is that some households never participate in financial markets. Occhino (2004) argues that the segmented markets model displays more persistence than other limited participation models, like the models in Grossman and Weiss (1983), Lucas (1990), Fuerst (1992), and Christiano and Eichenbaum (1992), where households do not participate in financial markets only temporarily. Also, Occhino (2003) shows that, differently from representative agents monetary models, the segmented markets model can replicate the persistent decrease of the aggregate output growth rate, and the persistent increase of the real interest rate, which follow a contractionary monetary policy shock.

The equilibrium of the segmented markets model, however, has been fully characterized only in the case that monetary policy is defined in terms of the money supply. This paper proves the existence of an equilibrium in the challenging case that monetary policy is defined in terms of the short-term nominal interest rate. The analysis is empirically important since monetary authorities of most OECD countries follow interest rate monetary policies. In the U.S. case, Bernanke and Blinder (1992) show that the federal funds rate, an overnight interest rate, is an excellent indicator of the stance of monetary policy, and innovations to the federal funds rate can be identified with monetary policy shocks.

The model allows the detrended aggregate output and the nominal interest rate to jointly follow any stationary process. It covers, then, the important cases where monetary policy affects output, and responds to any source of uncertainty, including output itself. The assumptions required for the existence of an equilibrium constrain the maximum value and the variability of the nominal interest rate. The period utility function is logarithmic.

The proof consists in the following steps. First, the equilibrium is characterized by a system of difference equations. Then, attention is restricted to recursive equilibria where all normalized variables depend in a time-invariant way on the aggregate state of the economy, and the equilibrium is characterized by a central functional equation. Finally, an operator is defined whose fixed point is a solution to the functional equation, and the existence of a fixed point is established. The proof involves showing

that the operator is monotone, which can be exploited to obtain a numerical solution. In addition, Occhino (2000) shows that a similar proof can be used in the case that monetary policy is defined in terms of the bond supply, like in Lucas (1990).

The paper is organized as follows. Section 2 describes the economy and defines the competitive equilibrium, Section 3 derives the central functional equation characterizing the equilibrium, and Section 4 proves the existence of a solution.

2 Economy

The model is a cash-in-advance endowment economy, with a large number of households and a monetary authority. Time is discrete and is indexed by $t \geq 0$. There are a single non-durable consumption good, money, and one-period nominal bonds, which are claims to one unit of money payable at the end of the period. Households are of two types, traders and non-traders. Let ω and ω^* be respectively the number of traders and non-traders. Households of the same type are identical in all respects. The crucial difference between the two types of households is that non-traders spend all their money purchasing consumption goods, while traders can purchase bonds as well.

Households start each period with cash balances from the previous period. Then, two markets meet in sequence, a bond market and a goods market. In the bond market, the monetary authority sells one-period nominal bonds to the traders, at the bond price $q_t > 0$. Open market operations are conducted in terms of the short-

term nominal interest rate. The monetary authority announces the bond price q_t and stands ready to issue and sell any number of bonds to clear the market at that price. By assumption, the interest rate is strictly positive, and the bond price is strictly less than one.

After the bond market, all households participate in the goods market. Traders and non-traders receive respectively constant fractions $\Lambda > 0$ and $\Lambda^* > 0$ of the exogenous stochastic aggregate endowment $Y_t > 0$, with $\omega\Lambda + \omega^*\Lambda^* = 1$. The endowment cannot be consumed directly, and must be sold in exchange of money at the price $P_t > 0$. Households can only consume goods purchased with money held before the goods market session. The money supply is defined as the amount of dollars $P_t Y_t$ spent in the goods market. Bonds are redeemed after the goods market closes.

The aggregate endowment Y_t and the bond price q_t are the only sources of uncertainty in the economy. Let $\{\bar{Y}_t\}_{t=0}^\infty$ be the non-stochastic steady state values of the aggregate endowment, and let us assume that \bar{Y}_{t+1}/\bar{Y}_t is constant and equal to α . We assume that the normalized aggregate endowment $y_t \equiv Y_t/\bar{Y}_t$ and the bond price q_t are time-invariant functions $y(z_t)$ and $q(z_t)$ of the exogenous state z_t of the economy. In turn, z_t follows a first-order Markov process with transition function $P : Z \times \mathcal{B}(Z) \rightarrow [0, 1]$, where Z is a compact Borel set, and $\mathcal{B}(Z)$ denotes the Borel subsets of Z . Intuitively, $P(z, A)$ is the probability that the next-period exogenous state belongs to the set $A \in \mathcal{B}(Z)$ given that the current exogenous state is equal to $z \in Z$. The exogenous state z_t is revealed at the very beginning of period t .

Each trader chooses consumption C_t , bonds B_t , and next-period cash balances A_{t+1} to solve

$$\max_{\{C_t > 0, B_t, A_{t+1} > 0\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t) \right]$$

subject to $q_t B_t + P_t C_t \leq A_t$

$$A_{t+1} = A_t - q_t B_t - P_t C_t + P_t \Lambda Y_t + B_t,$$

given the trader's initial cash balances $A_0 > 0$ in period zero, where E_0 is the expectation conditional on information available after z_0 has been revealed, the period utility function $u(C) \equiv \log(C)$ is logarithmic, and $\beta \in (0, 1)$.

Since the bond price q_t is strictly less than one for all t , holding idle cash balances is never optimal for traders, so the traders' cash-in-advance constraint always holds with equality. Then, the two constraints in the above maximization problem can be substituted with the constraints

$$q_t B_t + P_t C_t = A_t$$

$$A_{t+1} = P_t \Lambda Y_t + B_t$$

Non-traders spend all their initial cash balances purchasing consumption goods. Under this assumption, the behavior of a non-trader is simply described by

$$P_t C_t^* = A_t^*, \quad A_{t+1}^* = P_t \Lambda^* Y_t$$

given the non-traders' initial cash balances $A_0^* > 0$ in period zero.

The economy is described by the traders' initial assets $A_0 > 0$, the non-traders initial assets $A_0^* > 0$, the initial exogenous state z_0 , and the process for z_t . An equilibrium is a set of contingent sequences $\{C_t > 0, B_t, A_{t+1} > 0\}_{t=0}^{\infty}$ of consumption demand, bonds demand and cash balances for traders, $\{C_t^* > 0, A_{t+1}^* > 0\}_{t=0}^{\infty}$ of consumption demand and cash balances for non-traders, a contingent sequence $\{D_t\}_{t=0}^{\infty}$ of bonds supplied by the monetary authority, and a contingent sequence $\{P_t > 0\}_{t=0}^{\infty}$ of prices such that, given the prices, the traders' contingent sequence solves the traders' optimization problem, the non-traders' contingent sequence satisfies the non-traders equations, and the following bonds and goods market equilibrium condition hold:

$$\omega B_t = D_t$$

$$\omega C_t + \omega^* C_t^* = Y_t$$

The necessary first-order conditions for the traders' optimization problem are

$$\beta^t u'(C_t) - \nu_t^1 P_t = 0$$

$$-q_t \nu_t^1 + \nu_t^2 = 0$$

$$-\nu_t^2 + E_t[\nu_{t+1}] = 0$$

and the transversality condition is

$$\lim_{t \rightarrow \infty} E_0 [\nu_t^1 A_t] = 0$$

where ν_t^1 and ν_t^2 are the Lagrange multipliers associated with the two constraints. It follows that

$$\beta^t u'(C_t) = \nu_t^1 P_t$$

$$q_t \nu_t^1 = E_t[\nu_{t+1}^1]$$

The system describing the equilibrium is then

$$\beta^t / C_t = \nu_t^1 P_t$$

$$q_t \nu_t^1 = E_t[\nu_{t+1}^1]$$

$$q_t B_t + P_t C_t = A_t$$

$$A_{t+1} = P_t \Lambda Y_t + B_t$$

$$P_t C_t^* = A_t^*$$

$$A_{t+1}^* = P_t \Lambda^* Y_t$$

$$\omega B_t = D_t$$

$$\omega C_t + \omega^* C_t^* = Y_t$$

3 Solution

For convenience, variables are normalized as follows. As in Lucas and Stokey (1987) and Lucas (1990), nominal variables are normalized by aggregate cash balances available at the beginning of the period. Let $\bar{A}_t \equiv \omega A_t + \omega^* A_t^*$ be the initial aggregate cash balances. Then, $y_t \equiv Y_t / \bar{Y}_t$, $\mu_t \equiv \nu_t^1 \bar{A}_t / \beta^t \omega$, $c_t \equiv \omega C_t / \bar{Y}_t$, $b_t \equiv \omega B_t / \bar{A}_t$, $a_t \equiv \omega A_t / \bar{A}_t$, $c_t^* \equiv \omega^* C_t^* / \bar{Y}_t$, $a_t^* \equiv \omega^* A_t^* / \bar{A}_t$, $d_t \equiv D_t / \bar{A}_t$, $g_t \equiv \bar{A}_{t+1} / \bar{A}_t$, $p_t \equiv P_t \bar{Y}_t / \bar{A}_t$. Also, let us define $\lambda \equiv \omega \Lambda$, so $1 - \lambda = \omega^* \Lambda^*$. The previous system can then be written as

$$1/c_t = \mu_t p_t$$

$$q_t g_t \mu_t = \beta E_t[\mu_{t+1}]$$

$$q_t b_t + p_t c_t = a_t$$

$$g_t a_{t+1} = p_t \lambda y_t + b_t$$

$$p_t c_t^* = a_t^*$$

$$g_t a_{t+1}^* = p_t (1 - \lambda) y_t$$

$$b_t = d_t$$

$$c_t + c_t^* = y_t$$

$$a_t + a_t^* = 1$$

and the transversality condition can be written as

$$\lim_{t \rightarrow \infty} E_0 [\beta^t \mu_t a_t] = 0$$

It is convenient to derive the following two equations. From the goods market equilibrium condition, the non-traders' cash-in-advance constraint, and the last equation of the system, it follows that

$$p_t c_t + p_t c_t^* = p_t y_t$$

$$p_t c_t + a_t^* = p_t y_t$$

$$p_t c_t + (1 - a_t) = p_t y_t$$

Also, from the households' budget constraints, and the last equation of the system, it follows that

$$g_t a_{t+1} + g_t a_{t+1}^* = p_t \lambda y_t + b_t + p_t (1 - \lambda) y_t$$

$$g_t = p_t y_t + b_t$$

Then, using the equation just derived and the traders' cash-in-advance constraint,

$$g_t = p_t c_t + (1 - a_t) + b_t$$

$$g_t = 1 - q_t b_t + b_t$$

The equilibrium is, then, characterized by the following equivalent system

$$1/c_t = \mu_t p_t$$

$$q_t g_t \mu_t = \beta E_t[\mu_{t+1}]$$

$$q_t b_t + p_t c_t = a_t$$

$$g_t a_{t+1} = p_t \lambda y_t + b_t$$

$$g_t = 1 - q_t b_t + b_t$$

$$p_t c_t + (1 - a_t) = p_t y_t$$

$$p_t c_t^* = a_t^*$$

$$a_t + a_t^* = 1$$

$$b_t = d_t$$

Notice that the first six equations of the system do not depend on the variables c_t^* , a_t^* and d_t . Also, with a solution to the first part of the system, it is easy to determine c_t^* , a_t^* and d_t from the last three equations. We now look, then, for a solution to the first part of the system.

Let us restrict attention to stationary recursive equilibria, where aggregate normalized variables depend in a time-invariant way on two aggregate state variables. The first state variable is the exogenous state z_t . The second state variable is the share $\theta_t \equiv a_t$ of cash balances held by traders at the beginning of the period. Let us denote $\Theta \subseteq (0, 1]$ the interval where θ_t takes values in equilibrium. For convenience, $s_t \equiv (z_t, \theta_t)$ is the aggregate state of the economy, and $S \equiv Z \times \Theta$ is the set of possible values that s_t can take. The evolution of θ_t must be determined endogenously. In particular, one must determine an interval Θ , and a law of motion $\theta_{t+1} = \theta'(s_t)$, such that $\theta'(s_t) \in \Theta$, for all $s_t \in S$.

Recall that the bond price q_t and the normalized aggregate endowment y_t are known functions of the exogenous state z_t . Making explicit the dependence of the other variables on the aggregate state, the previous system becomes:

$$1/c(s_t) = \mu(s_t)p(s_t)$$

$$q(z_t)g(s_t)\mu(s_t) = \beta E[\mu(z_{t+1}, \theta'(s_t))|z_t] = \beta \int_Z \mu(z_{t+1}, \theta'(s_t))P(z_t, dz_{t+1})$$

$$q(z_t)b(s_t) + p(s_t)c(s_t) = \theta_t$$

$$g(s_t)\theta'(s_t) = p(s_t)\lambda y(z_t) + b(s_t)$$

$$g(s_t) = 1 - q(z_t)b(s_t) + b(s_t)$$

$$p(s_t)c(s_t) + (1 - \theta_t) = p(s_t)y(z_t)$$

In what follows, we will establish the existence of a state space Θ and a solution to the previous system such that $\theta'(s)$ belongs to Θ . The equilibrium can be, then,

easily derived. Notice that, since $\beta \in (0, 1)$, the transversality condition

$$\lim_{t \rightarrow \infty} E_0 \left[\beta^t \mu(s_t) a(s_t) \right] = 0$$

is satisfied in any recursive stationary equilibrium.

For convenience, let us drop the time index. Let us start deriving a preliminary equation for the law of motion $\theta'(s)$. From the traders' cash-in-advance constraint, one obtains

$$q(z)b(s) + p(s)c(s) = \theta$$

$$1 - \theta + p(s)c(s) = 1 - q(z)b(s)$$

$$p(s)y(z) = 1 - q(z)b(s),$$

where the last step uses the goods market equilibrium condition. Then, from the traders' budget constraint

$$\theta'(s)g(s) = b(s) + p(s)\lambda y(z)$$

$$\theta'(s)g(s) = (1 - \lambda)b(s) + \lambda[p(s)y(z) + b(s)]$$

$$\theta'(s)g(s) = (1 - \lambda)b(s) + \lambda[1 - q(z)b(s) + b(s)]$$

$$\theta'(s)[1 - q(z)b(s) + b(s)] = (1 - \lambda)b(s) + \lambda[1 - q(z)b(s) + b(s)]$$

$$\theta'(s) = \lambda + (1 - \lambda)b(s)/[1 - q(z)b(s) + b(s)]$$

where the second step follows from the preliminary equation previously derived, and the third from the expression for $g(s)$. After defining the value of the bond supply

$v(s) \equiv q(z)b(s)$, the previous relation can be written as

$$\theta'(s) = \lambda + (1 - \lambda) \frac{v(s)}{q(z) - q(z)v(s) + v(s)}. \quad (1)$$

We are, now, in a position to derive a central functional equation for the value of the bond supply $v(s)$. From the traders' first-order condition, one obtains

$$\begin{aligned} \beta \int_{\mathcal{Z}} \mu(z', \theta'(s)) P(z, dz') &= \mu(s) q(z) g(s) \\ \beta \int_{\mathcal{Z}} \frac{1}{p(z', \theta'(s)) c(z', \theta'(s))} P(z, dz') &= \frac{q(z) g(s)}{p(s) c(s)}. \\ \beta \int_{\mathcal{Z}} \frac{1}{\theta'(s) - q(z') b(z', \theta'(s))} P(z, dz') &= \frac{q(z) [1 - q(z) b(s) + b(s)]}{\theta - q(z) b(s)}, \end{aligned}$$

where the last step uses the expression for $g(s)$, and the traders' cash-in-advance constraint. Substituting $v(s) \equiv q(z)b(s)$, and using the previous relation 1, one obtains the functional equation

$$\frac{v(s)}{\theta - v(s)} \equiv \int_{\mathcal{Z}} \frac{\beta}{1 - \lambda} \frac{\theta'(s) - \lambda}{\theta'(s) - v(z', \theta'(s))} P(z, dz'), \quad (2)$$

all $s \in S$, where $s \equiv (z, \theta)$, and the law of motion $\theta'(s)$ satisfies the relation 1.

4 Existence

This section establishes the existence of a space Θ where θ takes values in equilibrium, and a solution $v(s)$ to the functional equation 2, such that the function $\theta'(s)$ given by the relation 1 takes values in Θ , for all $s \in S$. With the solution in hand, one can, then, derive the equilibrium.

The assumptions required constrain the level and the variability of the nominal interest rate. The proof consists in defining an operator whose fixed point is a solution to the functional equation, showing that the operator is monotone, using the monotonicity of the operator to construct a sequence of functions converging to a limit point, and showing that the limit point is, indeed, a fixed point of the operator. The proof, however, is complicated by the fact that the unknown function appears as argument of itself in the functional equation. It is, then, necessary to work with a subset of functions with known monotonicity properties, and to establish that the operator maps that set into itself.

First, let us introduce the function $v^*(q)$, $v^* : (0, \beta) \rightarrow (0, \beta)$, as follows

$$v^*(q) \equiv \frac{\beta - q}{1 - q}.$$

$v^*(q)$ is equal to β when q is equal to 0, is continuous and strictly decreasing in q , and is equal to 0 when q is equal to β .

$v^*(q)$ has the following economic interpretation. Consider for a moment a deterministic stationary economy where the bond price is constant and equal to q , the value of the bond supply is constant and equal to v , all the normalized variables are constant, and all the nominal variables grow at the growth rate $1 - v + v/q$ of the aggregate cash balances. Since the multiplier is constant over time, from the traders' Euler equation it follows that $q(1 - v + v/q) = \beta$, which is simply the Fisher equation stating that the gross real interest rate is equal to the ratio of the gross nominal interest rate to the gross inflation rate. The equation can be written as $q - qv + v = \beta$,

or $v = (\beta - q)/(1 - q)$, so $v = v^*(q)$. Hence, $v^*(q)$ is the constant value of the bond supply in a deterministic stationary economy where the bond price is constant and equal to q , all the normalized variables are constant, and all the nominal variables grow at the growth rate of the aggregate cash balances.

Let \underline{q} and \bar{q} be respectively the smallest and greatest values of q , and let us define $\bar{v} \equiv v^*(\bar{q})$, and $\underline{v} \equiv v^*(\underline{q})$. Let \mathcal{M} be the metric space of measurable functions on S taking values in $[\underline{v}, \bar{v}]$, with the sup norm. The following theorem 4.1 proves that, if $v(s)$ belongs to \mathcal{M} , then $\theta'(s)$ takes values in the interval $[\underline{\theta}, \bar{\theta}]$, where $\underline{\theta} \equiv \lambda + (1 - \lambda)\underline{v}/(\bar{q} - \bar{q}\underline{v} + \underline{v})$, and $\bar{\theta} \equiv \lambda + (1 - \lambda)\bar{v}/(\underline{q} - \underline{q}\bar{v} + \bar{v})$. Then, after letting the interval Θ where θ takes values in equilibrium be the interval $[\underline{\theta}, \bar{\theta}]$, we will only need to determine a solution $v \in \mathcal{M}$ to the functional equation 2.

Theorem 4.1 *For any $v \in \mathcal{M}$, the function $\theta'(z, \theta)$ defined in 1 takes values in the interval $[\underline{\theta}, \bar{\theta}]$.*

Proof. For any $v \in \mathcal{M}$ and any $s \in S$,

$$\begin{aligned} \theta'(s) &= \lambda + (1 - \lambda) \frac{v(s)}{q(z) - q(z)v(s) + v(s)} \\ &\leq \lambda + (1 - \lambda) \frac{v(s)}{q(z) - q(z)\bar{v} + v(s)} \\ &\leq \lambda + (1 - \lambda) \frac{\bar{v}}{q(z) - q(z)\bar{v} + \bar{v}} \\ &\leq \lambda + (1 - \lambda) \frac{\bar{v}}{\underline{q} - \underline{q}\bar{v} + \bar{v}} = \bar{\theta}; \end{aligned}$$

where the first inequality follows from $v(s) \leq \bar{v}$ for any $s \in S$; the second from $\bar{v} < 1$ and $v(s) \leq \bar{v}$ for any $s \in S$; the third from $\bar{v} < 1$ and $q(z) \geq \underline{q}$ for any $s \in S$; and the last equality from the definition of $\bar{\theta}$.

Similarly, one can show that $\theta'(s) \geq \underline{\theta}$. ■

To determine a solution to the functional equation 2, let us make the following assumption constraining the minimum value \underline{q} of the process for $q(z)$ relative to the level of markets segmentation:

Assumption 4.1 $\bar{v} \leq \lambda$.

The assumption constrains the ratio of the value of the bond supply to the sum of the value of the bond supply and the money supply to be less than the traders' share of the aggregate endowment.

Let us define the operator T as follows:

$$(Tv)(s) \equiv \frac{\theta R(s)}{1 + R(s)}, \quad (3)$$

where $R(s)$ is the right hand side of the functional equation 2. The following theorem 4.2 proves that, under assumption 4.1, $T : \mathcal{M} \rightarrow \mathcal{M}$. Since a fixed point of T is a solution to the functional equation 2, we will need to look for a fixed point of T .

Theorem 4.2 *Under assumption 4.1, the operator T defined in 3 satisfies $T : \mathcal{M} \rightarrow \mathcal{M}$.*

Proof. For any $v \in \mathcal{M}$ and any $s \in S$, the right hand side of the functional

equation 2 is

$$\begin{aligned}
R(s) &\equiv \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'(s) - \lambda}{\theta'(s) - v(z', \theta'(s))} P(z, dz') \\
&\leq \frac{\beta}{1 - \lambda} \frac{\theta'(s) - \lambda}{\theta'(s) - \bar{v}} \\
&\leq \frac{\beta}{1 - \lambda} \frac{\bar{\theta} - \lambda}{\bar{\theta} - \bar{v}} \\
&= \frac{\beta \bar{v}}{q - q\bar{v} + \bar{v}} \frac{1}{\bar{\theta} - \bar{v}} \\
&= \frac{\bar{v}}{\bar{\theta} - \bar{v}} \equiv \bar{R};
\end{aligned}$$

where the first inequality follows from $v(s) \leq \bar{v}$ for any $s \in S$; the second from assumption 4.1 and theorem 4.1; the next equality from the definition of $\bar{\theta}$; and the following from the definition of \bar{v} . Hence,

$$(Tv)(s) = \frac{\theta R(s)}{1 + R(s)} \leq \frac{\bar{\theta} \bar{R}}{1 + \bar{R}} = \frac{\bar{\theta} \bar{v}}{\bar{\theta} - \bar{v}} \frac{\bar{\theta} - \bar{v}}{\bar{\theta}} = \bar{v};$$

where the first equality follows from the definition of T ; the inequality from $\theta \leq \bar{\theta} < 1$ for any $\theta \in \Theta$, and from the inequality previously derived; and the next equality from the definition of \bar{R} .

Similarly, one can show that

$$R(s) \geq \frac{\underline{v}}{\underline{\theta} - \underline{v}} \equiv \underline{R},$$

and that

$$(Tv)(s) = \frac{\theta R(s)}{1 + R(s)} \geq \underline{v}.$$

The measurability of Tv follows from the fact that $P(z, A)$ is a measurable function of z for all $A \in \mathcal{B}(Z)$. ■

It is convenient to restrict our search for a fixed point of T to the subset $D \subset \mathcal{M}$ of the functions $v(s)$ such that $v(z, \theta)/\theta$ is weakly increasing in θ : as the traders' share of cash balances increases, their nominal investment in bonds increases relative to their consumption expenditure. Notice that, if $v \in D$, $v(z, \theta)$ is strictly increasing in θ : as the traders' share of cash balances increases, their nominal investment in bonds increases in absolute value. Then, the following theorem 4.3 proves that $\theta'(z, \theta)$ is strictly increasing in θ , theorem 4.4 proves that $T : D \rightarrow D$, and theorem 4.5 proves that T is monotone.

Theorem 4.3 *For any $v \in D$, the function $\theta'(z, \theta)$ defined in 1 is strictly increasing in θ .*

Proof. For any $v \in D$, any $z \in Z$ and any $\theta^1, \theta^2 \in \Theta$, $\theta^1 < \theta^2$,

$$\begin{aligned} \theta'(z, \theta^1) &= \lambda + (1 - \lambda) \frac{v(z, \theta^1)}{q(z) - q(z)v(z, \theta^1) + v(z, \theta^1)} \\ &< \lambda + (1 - \lambda) \frac{v(z, \theta^1)}{q(z) - q(z)v(z, \theta^2) + v(z, \theta^1)} \\ &< \lambda + (1 - \lambda) \frac{v(z, \theta^2)}{q(z) - q(z)v(z, \theta^2) + v(z, \theta^2)} = \theta'(z, \theta^2); \end{aligned}$$

where the first inequality follows from the fact that $v(z, \theta)$ is strictly increasing in θ ; and the second from $v(s) \leq \bar{v} < 1$ for any $s \in S$, and the fact that $v(z, \theta)$ is strictly increasing in θ . ■

Theorem 4.4 *Under assumption 4.1, the operator T defined in 3 satisfies $T : D \rightarrow D$.*

Proof. In light of theorem 4.2, I only need to show that $(Tv)(z, \theta)/\theta$ is weakly increasing in θ . Indeed, I will show that the monotonicity is strict. For any $v \in D$, any $z \in Z$ and any $\theta^1, \theta^2 \in \Theta$, $\theta^1 < \theta^2$, the right hand side of the functional equation 2 is

$$\begin{aligned} R(z, \theta^1) &\equiv \int_Z \frac{\beta}{(1-\lambda)} \frac{\theta'(z, \theta^1) - \lambda}{\theta'(z, \theta^1) - v(z', \theta'(z, \theta^1))} P(z, dz') \\ &< \int_Z \frac{\beta}{(1-\lambda)} \frac{\theta'(z, \theta^1) - \lambda}{\theta'(z, \theta^1) - v(z', \theta'(z, \theta^2))} P(z, dz') \\ &\leq \frac{\beta}{(1-\lambda)} \int_Z \frac{\theta'(z, \theta^2) - \lambda}{\theta'(z, \theta^2) - v(z', \theta'(z, \theta^2))} P(z, dz') = R(z, \theta^2); \end{aligned}$$

where the strict inequality follows from the fact that $v(z, \theta)$ is strictly increasing in θ and from theorem 4.3; and the weak inequality from $v(s) \leq \bar{v}$ for any $s \in S$, from assumption 4.1, and from theorem 4.3. It follows that

$$\frac{(Tv)(z, \theta^1)}{\theta^1} = \frac{R(z, \theta^1)}{1 + R(z, \theta^1)} < \frac{R(z, \theta^2)}{1 + R(z, \theta^2)} = \frac{(Tv)(z, \theta^2)}{\theta^2},$$

which concludes the proof that $(Tv)(z, \theta)/\theta$ is strictly increasing in θ . ■

Theorem 4.5 *Under assumption 4.1, the operator $T : D \rightarrow D$ defined in 3 is monotone.*

Proof. Consider any $v^1, v^2 \in D$ such that $v^1(s) \leq v^2(s)$ for any $s \in S$. Let us define $\theta^1(s)$ and $\theta^2(s)$ the function $\theta'(s)$ respectively when $v = v^1$ and $v = v^2$. Then, the following steps show that $\theta^1(s) \leq \theta^2(s)$ for any $s \in S$:

$$\begin{aligned} \theta^1(s) &= \lambda + (1-\lambda) \frac{v^1(s)}{q(z) - q(z)v^1(s) + v^1(s)} \\ &\leq \lambda + (1-\lambda) \frac{v^1(s)}{q(z) - q(z)v^2(s) + v^1(s)} \end{aligned}$$

$$\leq \lambda + (1 - \lambda) \frac{v^2(s)}{q(z) - q(z)v^2(s) + v^2(s)} = \theta'^2(s);$$

where the first inequality follows from $v^1(s) \leq v^2(s)$ for any $s \in S$; and the second from $v^2(s) \leq \bar{v} < 1$ for any $s \in S$, and $v^1(s) \leq v^2(s)$ for any $s \in S$.

Let us define $R^1(s)$ and $R^2(s)$ the right hand side of the functional equation 2 respectively when $v = v^1$ and $v = v^2$. Then, the following steps show that $R^1(s) \leq R^2(s)$ for any $s \in S$:

$$\begin{aligned} R^1(s) &= \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'^1(s) - \lambda}{\theta'^1(s) - v^1(z', \theta'^1(s))} P(z, dz') \\ &\leq \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'^1(s) - \lambda}{\theta'^1(s) - v^1(z', \theta'^2(s))} P(z, dz') \\ &\leq \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'^1(s) - \lambda}{\theta'^1(s) - v^2(z', \theta'^2(s))} P(z, dz') \\ &\leq \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'^2(s) - \lambda}{\theta'^2(s) - v^2(z', \theta'^2(s))} P(z, dz') = R^2(s); \end{aligned}$$

where the first inequality follows from $\theta'^1(s) \leq \theta'^2(s)$ for any $s \in S$, and from the fact that $v^2(z, \theta)$ is strictly increasing in θ ; the second from $v^1(s) \leq v^2(s)$ for any $s \in S$; and the third from $v^2(s) \leq \bar{v}$ for any $s \in S$, from assumption 4.1, and from $\theta'^1(s) \leq \theta'^2(s)$ for any $s \in S$.

It follows that, for any $s \in S$,

$$(Tv^1)(s) = \frac{\theta R^1(s)}{1 + R^1(s)} \leq \frac{\theta R^2(s)}{1 + R^2(s)} = (Tv^2)(s),$$

which concludes the proof that T is monotone. **■**

Now, let us construct a sequence of functions in D as follows:

$$v^0 \equiv \underline{v}, \text{ and } v^n \equiv T^n v^0, \text{ all } n \geq 1. \quad (4)$$

Notice that $T : D \rightarrow D$ implies that, if $v^n \in D$, then $v^{n+1} = Tv^n \in D$, all $n \geq 0$. Since $v^0 \in D$, by induction, $v^n \in D$, all $n \geq 0$.

Then, notice that $v^0 \equiv \underline{v}$ and $v^1 \in D$ imply $v^0 \leq v^1$. Also, the monotonicity of T implies that, if $v^n \leq v^{n+1}$, then $v^{n+1} = Tv^n \leq Tv^{n+1} = v^{n+2}$, all $n \geq 0$. Hence, by induction, the sequence $\{v^n\}_{n=0}^\infty$ is weakly increasing. Since $v^n \leq \bar{v}$, all $n \geq 0$, the sequence $\{v^n\}_{n=0}^\infty$ converges pointwise to a function, call it v^∞ . Since $v^n \in D$, all $n \geq 0$, and since v^∞ is the pointwise limit of the sequence $\{v^n\}_{n=0}^\infty$, v^∞ also belongs to D .

To compute a numerical approximation to v^∞ , Occhino (2000) discretizes the state space S with a grid of a finite number of states, applies the operator T to the constant function \underline{v} , and iterates until convergence is reached. Although v^∞ is not necessarily a fixed point of T , its numerical approximation turns out to be a fixed point of T and, therefore, a solution to the functional equation 2. Moreover, the same fixed point is obtained applying the operator T to the constant function \bar{v} and iterating until convergence is reached. Hence, abstracting from the fact that we are dealing with a numerical approximation of v^∞ and not with v^∞ itself, v^∞ is the only fixed point in D . The argument is the same as in the proof of the Corollary of Theorem 17.7 of Stokey and Lucas with Prescott (1989). Suppose that $v \in D$ is a fixed point of T . Notice that $v^0 \equiv \underline{v}$ and $v \in D$ imply $v^0 \leq v$. Also, the monotonicity of T implies that, if $v^n \leq v$, then $v^{n+1} = Tv^n \leq Tv = v$, all $n \geq 0$. Hence, by induction, $v^n \leq v$, all $n \geq 0$. Since v^∞ is the pointwise limit of the sequence $\{v^n\}_{n=0}^\infty$, $v^\infty \leq v$. A similar

argument starting with $v^0 \equiv \bar{v}$ leads to $v^\infty \geq v$. Hence, if $v \in D$ is a fixed point of T , then $v = v^\infty$.

An existence result is proven under the following additional assumption constraining the variability of the process for $q(z)$:

Assumption 4.2 $\bar{v}/\underline{v} \leq \beta/\bar{q}$.

Under this additional assumption, the following theorem 4.7 proves that $T : D_\theta \rightarrow D_\theta$, where D_θ is the subset of D of the functions v such that $\theta - v(z, \theta)$ is weakly increasing in θ : as the traders' share of cash balances increases, their consumption expenditure increases. Notice that any function $v(z, \theta)$ belonging to D_θ has bounded slope (and is, therefore, continuous) with respect to its second argument θ . As a corollary, the pointwise limit v^∞ of the sequence $\{v^n\}_{n=0}^\infty$ defined in 4 exists and belongs to D_θ . The next theorem 4.8 proves that v^∞ solves the functional equation 2.

Theorem 4.6 *For any $v \in D_\theta$, any $z \in Z$, and any $\theta^1, \theta^2 \in \Theta$, $\theta^1 < \theta^2$, the law of motion $\theta'(z, \theta)$ defined in 1 satisfies*

$$\theta'(z, \theta^2) - \theta'(z, \theta^1) \leq \frac{\theta'(z, \theta^1) - \lambda \bar{q}}{v(z, \theta^1)} \frac{\bar{q}}{\beta} [\theta^2 - \theta^1].$$

Proof. Consider any $v \in D_\theta$, any $z \in Z$, and any $\theta^1, \theta^2 \in \Theta$, $\theta^1 < \theta^2$. Then,

$$\begin{aligned} & \theta'(z, \theta^2) - \theta'(z, \theta^1) \\ &= (1 - \lambda) \left[\frac{v(z, \theta^2)}{q(z) - q(z)v(z, \theta^2) + v(z, \theta^2)} - \frac{v(z, \theta^1)}{q(z) - q(z)v(z, \theta^1) + v(z, \theta^1)} \right] \\ &= (1 - \lambda) \left[\frac{q(z)[v(z, \theta^2) - v(z, \theta^1)]}{[q(z) - q(z)v(z, \theta^2) + v(z, \theta^2)][q(z) - q(z)v(z, \theta^1) + v(z, \theta^1)]} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\theta'(z, \theta^1) - \lambda}{v(z, \theta^1)} \frac{q(z)[v(z, \theta^2) - v(z, \theta^1)]}{q(z) - q(z)v(z, \theta^2) + v(z, \theta^2)} \\
&\leq \frac{\theta'(z, \theta^1) - \lambda}{v(z, \theta^1)} \frac{q(z)[\theta^2 - \theta^1]}{q(z) - q(z)v(z, \theta^2) + v(z, \theta^2)} \\
&\leq \frac{\theta'(z, \theta^1) - \lambda}{v(z, \theta^1)} \frac{q(z)}{q(z) - q(z)\underline{v} + \underline{v}} [\theta^2 - \theta^1] \\
&\leq \frac{\theta'(z, \theta^1) - \lambda}{v(z, \theta^1)} \frac{\bar{q}}{\bar{q} - \bar{q}\underline{v} + \underline{v}} [\theta^2 - \theta^1] \\
&= \frac{\theta'(z, \theta^1) - \lambda}{v(z, \theta^1)} \frac{\bar{q}}{\beta} [\theta^2 - \theta^1];
\end{aligned}$$

where the first inequality follows from the fact that $\theta - v(z, \theta)$ is weakly increasing in θ ; the second from $q(z) < 1$ for any $z \in Z$ and from $v(s) \geq \underline{v}$ for any $s \in S$; the third from $q(z) \leq \bar{q}$ for any $z \in Z$; and the following equality from the definition of \underline{v} . ■

Theorem 4.7 *Under assumptions 4.1 and 4.2, the operator T defined in 3 satisfies $T : D_\theta \rightarrow D_\theta$.*

Proof. In light of the previous theorems, I only need to show that, if $v \in D_\theta$, then $\theta - (Tv)(z, \theta)$ is weakly increasing in θ . Consider any $v \in D_\theta$, any $z \in Z$, and any $\theta^1, \theta^2 \in \Theta$, $\theta^1 < \theta^2$. Then, the difference of the right hand side of the functional equation 2 evaluated respectively in $\theta = \theta^2$ and $\theta = \theta^1$ is

$$\begin{aligned}
&R(z, \theta^2) - R(z, \theta^1) \\
&\equiv \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'(z, \theta^2) - \lambda}{\theta'(z, \theta^2) - v(z', \theta'(z, \theta^2))} P(z, dz') \\
&\quad - \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'(z, \theta^1) - \lambda}{\theta'(z, \theta^1) - v(z', \theta'(z, \theta^1))} P(z, dz') \\
&\leq \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'(z, \theta^2) - \lambda}{\theta'(z, \theta^1) - v(z', \theta'(z, \theta^1))} P(z, dz')
\end{aligned}$$

$$\begin{aligned}
& - \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'(z, \theta^1) - \lambda}{\theta'(z, \theta^1) - v(z', \theta'(z, \theta^1))} P(z, dz') \\
& = \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'(z, \theta^2) - \theta'(z, \theta^1)}{\theta'(z, \theta^1) - v(z', \theta'(z, \theta^1))} P(z, dz') \\
& \leq \int_Z \frac{\beta}{1 - \lambda} \frac{\theta'(z, \theta^1) - \lambda}{\theta'(z, \theta^1) - v(z', \theta'(z, \theta^1))} P(z, dz') \frac{1}{v(z, \theta^1)} \frac{\bar{q}}{\beta} [\theta^2 - \theta^1] \\
& = R(z, \theta^1) \frac{1}{v(z, \theta^1)} \frac{\bar{q}}{\beta} [\theta^2 - \theta^1];
\end{aligned}$$

where the first inequality follows from theorem 4.3 and the fact that $\theta - v(z, \theta)$ is weakly increasing in θ ; the second inequality from theorem 4.6; and the last equality from the definition of $R(s)$.

Now, the following steps show that, as θ increases, the percentage increase in the function $1 + R(z, \theta)$ is less or equal than the percentage increase in θ itself:

$$\begin{aligned}
& \frac{(1 + R(z, \theta^2)) - (1 + R(z, \theta^1))}{1 + R(z, \theta^1)} \\
& \leq \frac{1}{1 + R(z, \theta^1)} R(z, \theta^1) \frac{1}{v(z, \theta^1)} \frac{\bar{q}}{\beta} [\theta^2 - \theta^1]; \\
& = \frac{\theta^1 R(z, \theta^1)}{1 + R(z, \theta^1)} \frac{1}{v(z, \theta^1)} \frac{\bar{q}}{\beta} \frac{\theta^2 - \theta^1}{\theta^1} \\
& = (Tv)(z, \theta^1) \frac{1}{v(z, \theta^1)} \frac{\bar{q}}{\beta} \frac{\theta^2 - \theta^1}{\theta^1} \\
& \leq \frac{\bar{v} \bar{q}}{\underline{v} \beta} \frac{\theta^2 - \theta^1}{\theta^1} \\
& \leq \frac{\theta^2 - \theta^1}{\theta^1};
\end{aligned}$$

where the first inequality follows from the inequality previously obtained; the second equality from the definition of T ; the following inequality from $v(s) \geq \underline{v}$ for all $s \in S$ and theorem 4.2; and the last inequality from assumption 4.2.

The previous inequality implies that

$$\frac{1 + R(z, \theta^2)}{1 + R(z, \theta^1)} \leq \frac{\theta^2}{\theta^1},$$

$$\frac{\theta^1}{1 + R(z, \theta^1)} \leq \frac{\theta^2}{1 + R(z, \theta^2)},$$

so $\theta/(1 + R(z, \theta))$ is weakly increasing in θ . Since the definition of T implies that

$$\theta - (Tv)(s) = \theta - \frac{\theta R(s)}{1 + R(s)} = \frac{\theta}{1 + R(s)},$$

it follows that $\theta - (Tv)(z, \theta)$ is also weakly increasing in θ . ■

Theorem 4.8 *Under assumptions 4.1 and 4.2, the pointwise limit $v^\infty \in D_\theta$ of the sequence $\{v^n\}_{n=0}^\infty$ defined in 4 solves the functional equation 2.*

Proof. Let us define $\theta^n(s)$ and $\theta'^\infty(s)$ the function $\theta'(s)$ respectively when $v = v^n$ and $v = v^\infty$. Since the sequence $\{v^n\}$ converges pointwise to v^∞ , the sequence $\{\theta^n\}$ converges pointwise to θ'^∞ .

Now, for any $s \in S$, any $z' \in Z$, and any $n \geq 0$,

$$|v^\infty(z', \theta'^\infty(s)) - v^n(z', \theta^n(s))|$$

$$\leq |v^\infty(z', \theta'^\infty(s)) - v^n(z', \theta'^\infty(s))| + |v^n(z', \theta'^\infty(s)) - v^n(z', \theta^n(s))|$$

$$\leq |v^\infty(z', \theta'^\infty(s)) - v^n(z', \theta'^\infty(s))| + |\theta'^\infty(s) - \theta^n(s)|,$$

where the last inequality follows from the facts that $v^n(z, \theta)$ is strictly increasing in θ , and $\theta - v^n(z, \theta)$ is weakly increasing in θ , so the absolute value of the slope of $v^n(z, \theta)$ with respect to θ is less than one. As $n \rightarrow \infty$, the first absolute value converges

to zero since the sequence $\{v^n\}$ converges pointwise to v^∞ , and the second absolute value converges to zero since the sequence $\{\theta^n\}$ converges pointwise to θ^∞ . Notice, here, that we need to exploit the fact that the slopes of the functions belonging to D_θ with respect to their second argument are uniformly bounded.

Let us define $f^n(s, z')$, $f^\infty(s, z')$ and $\bar{f}(s, z')$ the argument of the integral on the right hand side of the functional equation 2 respectively when $v = v^n$, $v = v^\infty$ and when v is constant and equal to \bar{v} . From the results obtained so far, it follows that the sequence $\{f^n\}_{n=0}^\infty$ converges pointwise to f^∞ . Also, $f^n \leq \bar{f}$, all $n \geq 0$, f^n are integrable, all $n \geq 0$, and \bar{f} is also integrable. By the Lebesgue Dominated Convergence Theorem, f^∞ is integrable, and its integral is equal to the limit of the integrals of f^n .

Let us define $R^n(s)$ and $R^\infty(s)$ the right hand side of the functional equation 2 respectively when $v = v^n$ and $v = v^\infty$. From the results obtained so far, it follows that the sequence $\{R^n\}_{n=0}^\infty$ converges pointwise to R^∞ . Hence, for any $s \in S$,

$$\begin{aligned} v^\infty(s) &= \lim_{n \rightarrow \infty} v^{n+1}(s) = \lim_{n \rightarrow \infty} (Tv^n)(s) \\ &= \lim_{n \rightarrow \infty} \frac{\theta R^n(s)}{1 + R^n(s)} = \frac{\theta R^\infty(s)}{1 + R^\infty(s)} = (Tv^\infty)(s), \end{aligned}$$

that is v^∞ solves the functional equation 2. Notice, here, that the fourth equality follows from the results previously obtained, and not from the continuity of T . The reason is that T is uniformly continuous in the sup norm, while the sequence $\{v^n\}$ converges to v^∞ only pointwise. **■**

References

- [1] Alvarez, F., Lucas, R. E. Jr., and Weber, W. (2001). “Interest Rate and Inflation,” *American Economic Review, (Papers and Proceedings)* **91(2)**, 219–225.
- [2] Bernanke, B. S., and Blinder, A. S. (1992). “The Federal Funds Rate and the Channels of Monetary Transmission,” *American Economic Review* **82(4)**, 901–921.
- [3] Christiano, L. J., and Eichenbaum, M. (1992). “Liquidity Effects and the Monetary Transmission Mechanism,” *American Economic Review (Papers and Proceedings)* **82(2)**, 346–353.
- [4] Fuerst, T. (1992). “Liquidity, Loanable Funds and Real Activity,” *Journal of Monetary Economics* **29(1)**, 3–24.
- [5] Grossman, S., and Weiss, L. (1983). “A Transactions-Based Model of the Monetary Transmission Mechanism,” *American Economic Review* **73(5)**, 871–880.
- [6] Lahiri, A., Singh, R., and Vegh, C. (2003). “Segmented Asset Markets and Optimal Exchange Rate Regimes,” Manuscript.
- [7] Lucas, R. E. Jr. (1990). “Liquidity and Interest Rates,” *Journal of Economic Theory* **50(2)**, 237–264.
- [8] Lucas, R. E. Jr., and Stokey, N. L. (1987). “Money and Interest in a Cash-in-Advance Economy,” *Econometrica* **55(3)**, 491–513.

- [9] Occhino, F. (2000). *Heterogeneous Investment Behavior and the Persistence of the Liquidity Effect*, Ph. D. Dissertation, The University of Chicago.
- [10] Occhino, F. (2003). “Markets Segmentation and the Response of the Real Interest Rate to Monetary Policy Shocks,” Manuscript available at the Web address <http://www.rci.rutgers.edu/~occhino/Research/realinterestrates.pdf>.
- [11] Occhino, F. (2004). “Modeling the Response of Money and Interest Rates to Monetary Policy Shocks: A Segmented Markets Approach,” *Review of Economic Dynamics* **7(1)**, 181–197.
- [12] Stokey, N. L., Lucas, R. E. Jr., and Prescott, E. C. (1989). *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, Massachusetts.