

Schlicht, Ekkehart

Working Paper

Estimating the Smoothing Parameter in the So-Called Hodrick-Prescott Filter

IZA Discussion Papers, No. 1054

Provided in Cooperation with:

IZA – Institute of Labor Economics

Suggested Citation: Schlicht, Ekkehart (2004) : Estimating the Smoothing Parameter in the So-Called Hodrick-Prescott Filter, IZA Discussion Papers, No. 1054, Institute for the Study of Labor (IZA), Bonn

This Version is available at:

<https://hdl.handle.net/10419/20289>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

IZA DP No. 1054

Estimating the Smoothing Parameter in the So-Called Hodrick-Prescott Filter

Ekkehart Schlicht

March 2004

Estimating the Smoothing Parameter in the So-Called Hodrick-Prescott Filter

Ekkehart Schlicht

*University of Munich
and IZA Bonn*

Discussion Paper No. 1054
March 2004

IZA

P.O. Box 7240
53072 Bonn
Germany

Phone: +49-228-3894-0
Fax: +49-228-3894-180
Email: iza@iza.org

Any opinions expressed here are those of the author(s) and not those of the institute. Research disseminated by IZA may include views on policy, but the institute itself takes no institutional policy positions.

The Institute for the Study of Labor (IZA) in Bonn is a local and virtual international research center and a place of communication between science, politics and business. IZA is an independent nonprofit company supported by Deutsche Post World Net. The center is associated with the University of Bonn and offers a stimulating research environment through its research networks, research support, and visitors and doctoral programs. IZA engages in (i) original and internationally competitive research in all fields of labor economics, (ii) development of policy concepts, and (iii) dissemination of research results and concepts to the interested public.

IZA Discussion Papers often represent preliminary work and are circulated to encourage discussion. Citation of such a paper should account for its provisional character. A revised version may be available on the IZA website (www.iza.org) or directly from the author.

ABSTRACT

Estimating the Smoothing Parameter in the So-Called Hodrick-Prescott Filter*

This note gives a fairly complete statistical description of the Hodrick-Prescott Filter (1997) which has been proposed in the context of my seasonal adjustment method (Schlicht 1981, 1984). A statistics estimator for the smoothing parameter is proposed that is asymptotically equivalent to the maximum-likelihood estimator and has a straightforward intuitive interpretation. The method is illustrated by an application and several simulations.

JEL Classification: C22

Keywords: Hodrick-Prescott filter, Kalman filtering, Kalman-Bucy, state-space models, random walk, time-varying coefficients, adaptive estimation, time-series, seasonal adjustment, trend

Ekkehart Schlicht
Institutional Economics Group
Department of Economics
University of Munich
Schackstr. 4
80539 Munich
Germany
Email: schlicht@lmu.de

* I thank Johannes Ludsteck for several helpful suggestions and for providing the Mathematica Package (Ludsteck, 2004) used for the computations in this paper.

INTRODUCTION

What is known as the Hodrick-Prescott Filter (1997) is widely used in applications and has been embodied in various statistical packages. The main problem is to determine an appropriate smoothing constant. The aim of this paper is to offer a rather comprehensive discussion of the appropriate maximum-likelihood estimator and a related and more intuitive statistics estimator.

It has been overlooked in the literature that the filter is part of my seasonal adjustment method (Schlicht 1981). Likewise, the statistical interpretation of the filter is covered by the statistical interpretation of my seasonal method (Schlicht 1984). The following remarks restate these findings and employ an estimation strategy for the present problem that has been used in Schlicht (1985, ch. 4; 1989) in the context of estimating time-varying coefficients models.

The paper is organized as follows. In Section 1 I describe the filter; in Section 2, I give a statistical interpretation that involves a formal "level parameter". Section 3 gives the estimator for that parameter. In Section 4 it is proven that the descriptive procedure described in Section 1 gives an unbiased maximum-likelihood estimate for the trend, given a smoothing parameter.

Given any smoothing parameter, the covariance matrix of the trend estimate is given in Section 5. Section 6 turns to estimation of the variances by a maximum likelihood method. The variances determine the smoothing parameter. It is shown that the numerical problem can be simplified considerably in several ways.

Section 7 describes another estimator for the variances. This estimator is characterized by the property that the computed variances of the error terms are equal to their expectations. In Section 8 it is shown that the likelihood estimates and the statistics estimates differ only slightly and approach each other with an increasing length of the time series. This gives intuitive appeal to the maximum likelihood estimator and statistical appeal to the statistics estimator.

Section 9 comments on some practical aspects and presents some simulations.

Section 10 comments on some additional statistical issues, and section 11 points out some open problems.

1. THE FILTER

Consider a time series $x \in \mathbb{R}^T$ that is to be decomposed into a trend $y \in \mathbb{R}^T$ and an irregular component $u \in \mathbb{R}^T$:

$$x = y + u \tag{1}$$

Define the trend disturbance $v \in \mathbb{R}^{T-1}$ as

$$v_t = ((y_t - y_{t-1}) - (y_{t-1} - y_{t-2}))^2 \quad t = 3, 4, \dots, T$$

or

$$v = P y \tag{2}$$

with

$$P := \begin{pmatrix} 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & & \cdot & \cdot & \cdot \\ 0 & & & 1 & -2 & 1 \end{pmatrix} \tag{3}$$

of order $(T - 2) \times T$.

The decomposition of the original series x into trend y and irregular component u is obtained by minimizing the weighted sum of squares

$$u'u + \alpha \cdot v'v = (x - y)'(x - y) + \alpha \cdot y'P'Py \tag{4}$$

with respect to y . This gives the first-order condition

$$(I + \alpha \cdot P'P) y = x \tag{5}$$

As $(I + \alpha P'P)$ is positive definite, the second order condition is satisfied in any case.

The the system matrix in (5) can be written as as

$$\left(I, \quad \alpha^{\frac{1}{2}} P' \right) \begin{pmatrix} I \\ \alpha^{\frac{1}{2}} P \end{pmatrix}$$

As $(I, \quad \alpha^{\frac{1}{2}} P')$ is of full rank, $(I + \alpha \cdot P'P)$ is nonsingular and equation (5) has the unique solution

$$y = (I + \alpha P'P)^{-1} x \tag{6}$$

Equation (6) defines the descriptive filter that associates a trend y to the time series x , depending on the smoothing parameter α .

2. STOCHASTIC INTERPRETATION

Equations (1) and (2) can be embedded in a stochastic model by assuming that the disturbances u and v in (1) and (2) are normal random variables with variances σ_u^2 and σ_v^2 and zero expectations:

$$u \sim \mathcal{N}(0, \sigma_u^2), \quad v \sim \mathcal{N}(0, \sigma_v^2) \quad (7)$$

This turns x and y into random variables with probability distributions that will be derived in the following.

As the rank of P is $T - 2$, there exist two orthogonal solutions z to the equation $Pz = 0$. These can be taken as columns of a $(T \times 2)$ -matrix Z that satisfies

$$PZ = 0, \quad Z'Z = 0 \quad (8)$$

The following observation will prove useful later on:

Claim.

$$\det(PP') = \frac{1}{12}T^2(T^2 - 1) \quad (9)$$

Proof. Checked with Mathematica for $T = 5$ to 2500. If $T > 2500$ are involved, the validity of this claim needs to be checked on a computer. \square

Equation (9) entails

$$\begin{aligned} \det\left(\begin{pmatrix} P'P & Z'Z \end{pmatrix}\right) &= \det\left(\begin{pmatrix} P \\ Z' \end{pmatrix} \begin{pmatrix} P' & Z \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} P \\ Z' \end{pmatrix} \begin{pmatrix} P' & Z \end{pmatrix}\right) \\ &= \det\begin{pmatrix} PP' & 0 \\ 0 & I \end{pmatrix} \\ &= \det(PP') \\ &= \frac{1}{12}T^2(T^2 - 1) \end{aligned} \quad (10)$$

Consider now

$$\begin{pmatrix} P \\ Z' \end{pmatrix} \begin{pmatrix} P' & Z \end{pmatrix} = \begin{pmatrix} PP' & 0 \\ 0 & I \end{pmatrix} \quad (11)$$

which is of full rank. Inverting both sides of (11), pre-multiplying by (P', Z) and multiplying from the right-hand side by $\begin{pmatrix} P \\ Z' \end{pmatrix}$ implies

$$\begin{pmatrix} P' & Z \end{pmatrix} \begin{pmatrix} (PP')^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} P \\ Z' \end{pmatrix} = I \quad (12)$$

and hence

$$P' (PP')^{-1} P + ZZ' = I \quad (13)$$

The auxiliary matrix Z comprises the set of the two orthogonal solutions to the equation $Pz = 0$. Any solution to (2) can be written as

$$y = P' (PP')^{-1} v + Z \beta \quad (14)$$

with $\beta \in \mathfrak{R}^2$ as a formal parameter vector. The time series x can be seen as the sum of y and u , and hence as brought about by the disturbances u and v :

$$x = u + P' (PP')^{-1} v + Z \beta \quad (15)$$

Combining (14) and (15) gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I & P' (PP')^{-1} \\ 0 & P' (PP')^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} Z \\ Z \end{pmatrix} \beta \quad (16)$$

As the disturbances u and v are normal with variances σ_u^2 and σ_v^2 , the vector (u', v') is normal as well:

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \mathcal{N}(0, S_{uv}) \quad (17)$$

The co-variance matrix is

$$S_{uv} := \begin{pmatrix} \sigma_u^2 \cdot I & 0 \\ 0 & \sigma_v^2 \cdot I \end{pmatrix} \quad (18)$$

From (16) to (18) we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} Z \\ Z \end{pmatrix} \beta, S_{xy} \right) \quad (19)$$

with¹

$$S_{xy} := \begin{pmatrix} \sigma_u^2 I + \sigma_v^2 Q & \sigma_v^2 Q \\ \sigma_v^2 Q & \sigma_v^2 Q \end{pmatrix} \quad (20)$$

and

$$Q := P' (PP')^{-1} (PP')^{-1} P \quad (21)$$

Note that (8) entails

$$Z'Q = 0 \quad (22)$$

From (19) we obtain the marginal density of x as

$$x \sim \mathcal{N}(Z\beta, S_x) \quad (23)$$

with

$$\begin{aligned} S_x &:= (\sigma_u^2 I + \sigma_v^2 Q) \\ &= \sigma_v^2 \left(I + \frac{\sigma_v^2}{\sigma_u^2} Q \right) \end{aligned} \quad (24)$$

Note that

$$\begin{aligned} Z'S_x^{-1} &= \frac{1}{\sigma_v^2} Z' \left(I - \frac{\sigma_v^2}{\sigma_u^2} Q + \left(\frac{\sigma_v^2}{\sigma_u^2} Q \right)^2 - \left(\frac{\sigma_v^2}{\sigma_u^2} Q \right)^3 + \left(\frac{\sigma_v^2}{\sigma_u^2} Q \right)^4 - \dots \right) \\ &= \frac{1}{\sigma_v^2} Z' \end{aligned} \quad (25)$$

The marginal density of y for given x is

$$(y|x) \sim \mathcal{N}(\bar{y}, S_y) \quad (26)$$

where

$$\bar{y} := Z\beta + \sigma_v^2 Q (\sigma_u^2 I + \sigma_v^2 Q)^{-1} (x - Z\beta) \quad (27)$$

$$S_{y|x} := \sigma_v^2 Q - \sigma_v^2 Q (\sigma_u^2 I + \sigma_v^2 Q)^{-1} \sigma_v^2 Q \quad (28)$$

Equations (27) and (28) can be simplified. Note that

$$I - \sigma_v^2 (\sigma_u^2 I + \sigma_v^2 Q)^{-1} Q = \sigma_u^2 (\sigma_u^2 I + \sigma_v^2 Q)^{-1}$$

¹Note that the covariance matrix S_{xy} is not of full rank. Hence (x', y') is distributed on a subspace of \mathfrak{R}^{2T} that is determined by the parameter λ .

which is verified by pre-multiplication with $(\sigma_u^2 I + \sigma_v^2 Q)$. Hence the covariance matrix can be written as

$$S_{y|x} = \sigma_u^2 \sigma_v^2 Q (\sigma_u^2 I + \sigma_v^2 Q)^{-1}$$

With $\alpha = \sigma_u^2 / \sigma_v^2$ we have

$$\sigma_v^2 Q (\sigma_u^2 I + \sigma_v^2 Q)^{-1} = Q (\alpha I + Q)^{-1} \quad (29)$$

and with (13)

$$\begin{aligned} (\alpha I + Q) (P' P) &= (\alpha P' P + P' (P P')^{-1} P) \\ &= (I + \alpha P' P - Z Z') \end{aligned}$$

Hence

$$(\alpha I + Q)^{-1} = (P' P) (I + \alpha P' P - Z Z')^{-1}$$

and

$$\sigma_v^2 Q (\sigma_u^2 I + \sigma_v^2 Q)^{-1} = (I - Z Z') (I + \alpha P' P - Z Z')^{-1} \quad (30)$$

With (29), and (30), (27) can be re-written as

$$\bar{y} = Z \beta + Q (\alpha I + Q)^{-1} (x - Z \beta) \quad (31)$$

or

$$\bar{y} = Z \beta + (I - Z Z') \left(I + \frac{\sigma_u^2}{\sigma_v^2} P' P - Z Z' \right)^{-1} (x - Z \beta) \quad (32)$$

Further we have

$$\begin{aligned} P' P Q &= \frac{\sigma_u^2}{\sigma_v^2} P' P + P' (P P')^{-1} P \\ &= I + \frac{\sigma_u^2}{\sigma_v^2} P' P - Z Z' \end{aligned}$$

and hence

$$S_{y|x} = \sigma_u^2 Q \left(I + \frac{\sigma_u^2}{\sigma_v^2} Q \right)^{-1} \quad (33)$$

or

$$S_{y|x} = \left(I - (I - Z Z') \left(I + \frac{\sigma_u^2}{\sigma_v^2} P' P - Z Z' \right)^{-1} \right) \sigma_v^2 Q \quad (34)$$

3. ESTIMATING THE FORMAL PARAMETERS β

The parameters that need to be estimated are the formal parameter vector β and the variances σ_u^2 and σ_v^2 .

The estimation of the formal parameters β is straightforward. Equation (23) gives rise to the likelihood function

$$L(x, \beta, \sigma_u^2, \sigma_v^2) : = -\log \det(\sigma_u^2 I + \sigma_v^2 Q) - (x - Z\beta)' (\sigma_u^2 I + \sigma_v^2 Q)^{-1} (x - Z\beta) \quad (35)$$

Minimizing L with respect to β leads to sufficient condition

$$Z' (\sigma_u^2 I + \sigma_v^2 Q)^{-1} x = Z' (\sigma_u^2 I + \sigma_v^2 Q)^{-1} Z \hat{\beta} \quad (36)$$

In view of (24), (25), and (8) this reduces to

$$\hat{\beta} = Z' x \quad (37)$$

4. ESTIMATING THE TREND y

If we substitute the formal parameters β with the estimator $\hat{\beta}$ in (37), we obtain the conditional distribution of the trend y (which is a random variable). It seems sensible to take the expectation of this random variable as our estimator for the trend. This yields:

$$\hat{y} := Z \hat{\beta} + Q \left(\frac{\sigma_u^2}{\sigma_v^2} I + Q \right)^{-1} (x - Z \hat{\beta}) \quad (38)$$

Theorem 1. *With the smoothing constant α equal to the variance ratio σ_u^2 / σ_v^2 the descriptive decomposition (6) is numerically identical to the estimator (38).*

Proof. Setting $\alpha = \sigma_u^2 / \sigma_v^2$ in (38) and ordering terms gives

$$\hat{y} = Z \hat{\beta} + Q (\alpha I + Q)^{-1} (x - Z \hat{\beta}) \quad (39)$$

Note that

$$Q (\alpha I + Q)^{-1} = I - \alpha (\alpha I + Q)^{-1} \quad (40)$$

which is verified by right-hand multiplication with $(\alpha I + Q)$. Inserting this into (39) and re-arranging terms gives

$$(\alpha I + Q)^{-1} (x - Z \hat{\beta}) = \frac{1}{\alpha} (x - \hat{y}) \quad (41)$$

This can be inserted into (39) again, and we obtain

$$\hat{y} = Z\hat{\beta} + Q\frac{1}{\alpha}(x - \hat{y}) \quad (42)$$

Pre-multiplication with $\alpha P'P$ yields

$$\alpha P'P\hat{y} = \alpha P'PZ\hat{\beta} + P'PQ(x - \hat{y}) \quad (43)$$

As $PZ = 0$, the first term on the right-hand side cancels. From the definition (21) of Q and (13) it follows that

$$P'PQ = I - ZZ' \quad (44)$$

Substituting this into (43) gives

$$\alpha P'P\hat{y} = (x - \hat{y}) - ZZ'(x - \hat{y}) \quad (45)$$

Because of (37) we have $Z'x = \hat{\beta}$. Pre-multiplying (42) by Z' while noting that $Z'Z = I$ and $Z'Q = 0$ results in $Z'\hat{y} = \hat{\beta}$ as well. Hence the last term in (45) cancels and we obtain

$$(I + \alpha P'P)\hat{y} = x \quad (46)$$

which is numerically identical to the normal equation (5) that defines the descriptive filter. \square

5. THE COVARIANCE MATRIX OF THE ESTIMATES

Consider a given time series x and a realization of the associated trend y . Because x can be viewed as brought about as the sum of the trend y and the disturbance u , we can write :

$$\hat{y} = (I + \alpha P'P)^{-1}(y + u) \quad (47)$$

Since

$$y = (I + \alpha P'P)y - \alpha P'Py \quad (48)$$

and $v = Py$, equation (47) can be written as

$$\hat{y} - y = (I + \alpha P'P)^{-1}(u - \alpha P'v) \quad (49)$$

Equation (49) gives the estimation error, and the covariance matrix of this error is calculated as

$$E\{(\hat{y} - y)'(\hat{y} - y)\} = \sigma_u^2(I + \alpha P'P)^{-1} \quad (50)$$

For given variances (and therefore a given smoothing constant $\alpha = \sigma_u^2 / \sigma_v^2$), equation (50) gives the variances of the trend estimates. The square roots of the main diagonal elements of (50) give the standard errors of the corresponding point estimates \hat{y}_t of the trend. It is thus possible to guess, for any smoothing parameter α , the precision of the trend estimate.

6. MAXIMUM-LIKELIHOOD ESTIMATION OF THE VARIANCES

In order to estimate the smoothing parameter α , we turn now to estimating the variances σ_u^2 and σ_v^2 . A first approach is to simply write down the maximum likelihood function. The distribution of the observations x is given by density function (23). Taking logarithms and disregarding constants gives the likelihood

$$L(x, \lambda, \sigma_u^2, \sigma_v^2) := -\log \det(\sigma_u^2 I + \sigma_v^2 Q) - (x - Z\beta)' (\sigma_u^2 I + \sigma_v^2 Q)^{-1} (x - Z\beta) \quad (51)$$

By replacing the parameter λ with its estimate $\hat{\lambda} = Z'x$ from (37), we obtain the concentrated likelihood

$$L^*(x, \sigma_u^2, \sigma_v^2) := -\log \det(\sigma_u^2 I + \sigma_v^2 Q) - x' (I - ZZ')' (\sigma_u^2 I + \sigma_v^2 Q)^{-1} (I - ZZ') x \quad (52)$$

This would suffice, in principle, to estimate the variances σ_u^2 and σ_v^2 , but the problem can be simplified considerably. The following theorem states that the likelihood (52) can be expressed in terms of the estimated trend \hat{y} and the weighted sum of the variances of the estimates errors \hat{u} and \hat{v} which are defined as follows:

$$\hat{y} := (I - \alpha P'P)^{-1} x \quad (53)$$

$$\hat{u} := x - \hat{y} \quad (54)$$

$$\hat{v} := P\hat{y} \quad (55)$$

Theorem 2. *The likelihood (52) can be written as*

$$L^*(x, \sigma_u^2, \sigma_v^2) = -\log \det(\sigma_u^2 I + \sigma_v^2 Q) - \frac{1}{\sigma_u^2} \hat{u}' \hat{u} - \frac{1}{\sigma_v^2} \hat{v}' \hat{v} \quad (56)$$

Proof. As the first terms of eq (52) and (56) are identical, it suffices to show that the quadratic forms in these equations are the same. Consider first the

quadratic in (56). From (53) we obtain

$$\hat{u}'\hat{u} = x' \left(I - (I + \alpha P'P)^{-1} \right) \left(I - (I + \alpha P'P)^{-1} \right) x \quad (57)$$

and

$$\hat{v}'\hat{v} = x' (I + \alpha P'P)^{-1} P'P (I + \alpha P'P)^{-1} x \quad (58)$$

Because

$$(I + \alpha P'P)^{-1} = I - \alpha P'P + (\alpha P'P)^2 - (\alpha P'P)^3 + \dots$$

the matrices $P'P$ and $(I + \alpha P'P)^{-1}$ commute and we can re-write equation (58) as

$$\hat{v}'\hat{v} = x' P'P (I + \alpha P'P)^{-1} (I + \alpha P'P)^{-1} x \quad (59)$$

Combining (56) and (58) gives

$$\hat{u}'\hat{u} + \alpha \hat{v}'\hat{v} = x' \left(I - (I + \alpha P'P)^{-1} \right) x \quad (60)$$

With

$$A := \left(I - (I + \alpha P'P)^{-1} \right) \quad (61)$$

and $\alpha = \sigma_u^2 / \sigma_v^2$ the quadratic in equation (56) is

$$\frac{1}{\sigma_u^2} \hat{u}'\hat{u} + \frac{1}{\sigma_v^2} \hat{v}'\hat{v} = \frac{1}{\sigma_u^2} x' A x \quad (62)$$

Consider next the quadratic in (52). With

$$B := (I - ZZ')' (\alpha I + Q)^{-1} (I - ZZ') \quad (63)$$

it is

$$x' (I - ZZ')' (\sigma_u^2 I + \sigma_v^2 Q)^{-1} (I - ZZ') x = \frac{1}{\sigma_v^2} x' B x \quad (64)$$

Right-hand multiplication of (61) by the non-singular matrices $(I + \alpha P'P)$ and $(\alpha I + Q)$ and use of (13) results in

$$A (I + \alpha P'P) (\alpha I + Q) = \alpha (I + \alpha P'P - ZZ') \quad (65)$$

Equation (63) can be re-written as

$$B = (\alpha I + Q)^{-1} - \frac{1}{\alpha} ZZ' \quad (66)$$

This makes use of the fact that the matrices $(I - ZZ')$ and $(\alpha I + Q)^{-1}$ commute, that $(I - ZZ')$ is idempotent and that $Z'Q = 0$. Right-hand multiplication of (66) by $(\alpha I + Q)$ and $(I + \alpha P'P)$ yields

$$B(I + \alpha P'P)(\alpha I + Q) = I + \alpha P'P - ZZ' \quad (67)$$

This makes use of the fact that the non-singular matrices $(I + \alpha P'P)$ and $(\alpha I + Q)$ commute and that $PZ = 0$. Equations (65) and (67) imply

$$\frac{1}{\sigma_u^2} A = \frac{1}{\sigma_v^2} B \quad (68)$$

Therefore the expressions given in equations (62) and (64) are identical. \square

For purposes of estimation, it is useful to parametrize the likelihood function (56) by α and σ_u^2 instead of σ_u^2 and σ_v^2 . Because $\sigma_v^2 = \sigma_u^2 / \alpha$, we can write:

$$\begin{aligned} L^{**}(x, \sigma_u^2, \alpha) &:= -\log \det(\alpha I + Q) - \frac{1}{\sigma_u^2} (\hat{u}'\hat{u} + \alpha \hat{v}'\hat{v}) \\ &\quad + T \cdot \log \alpha - T \cdot \log \sigma_u^2 \end{aligned} \quad (69)$$

For any given α , the minimization of L^{**} with respect to σ_u^2 leads to the necessary and sufficient conditions

$$\frac{\partial L^{**}}{\partial \sigma_u^2} = -\frac{T}{\sigma_u^2} + \frac{1}{\sigma_u^4} (\hat{u}'\hat{u} + \alpha \hat{v}'\hat{v}) = 0 \quad (70)$$

$$\left. \frac{\partial^2 L^{**}}{\partial (\sigma_u^2)^2} \right|_{\frac{\partial L^{**}}{\partial \sigma_u^2} = 0} = -\frac{T}{\sigma_u^4} < 0 \quad (71)$$

which imply the estimator

$$\hat{\sigma}_u^2 = \frac{1}{T} (\hat{u}'\hat{u} + \alpha \hat{v}'\hat{v}) \quad (72)$$

for the variance of u .

Given any smoothing parameter α , equation (58) permits estimating the precision of the trend estimates - the covariance matrix (50) - in terms of the calculated errors:

$$E \{(\hat{y} - y)' (\hat{y} - y)\} = \frac{1}{T} (\hat{u}'\hat{u} + \alpha \hat{v}'\hat{v}) (I + \alpha \cdot P'P)^{-1} \quad (73)$$

By inserting (72) into (6g) and disregarding constants, a concentrated likelihood function can be derived that involves the smoothing parameter α as its only parameter:

$$L^{***}(x; \alpha) := -\log \det(\alpha I + Q) - T \cdot \log R(\alpha) + T \cdot \log \alpha \quad (74)$$

with

$$R(\alpha) := \hat{u}'\hat{u} + \alpha\hat{v}'\hat{v} \quad (75)$$

With (74), maximum likelihood estimation reduces to maximizing over just one parameter. As the solution \hat{y} to the band-diagonal normal equation (6) is straightforward, minimization of L^{***} with respect to the smoothing parameter α can be performed numerically. The solution \hat{y} can be calculated for any α . The value of $R(\alpha)$ is calculated *via* (53) - (55) and (75). For any α , the corresponding variances are computed according to (72) and $\alpha = \sigma_u^2/\sigma_v^2$ as

$$\hat{\sigma}_u^2 = \frac{1}{T} R(\alpha) \quad (76)$$

$$\hat{\sigma}_v^2 = \frac{1}{T} \frac{R(\alpha)}{\alpha} \quad (77)$$

The likelihood function can be further simplified with respect to the first term. Consider

$$(\alpha I + Q) \begin{pmatrix} P' & Z \end{pmatrix} \begin{pmatrix} P \\ Z' \end{pmatrix} = (\alpha P'P + \alpha ZZ' + I - ZZ') \quad (78)$$

From (g) we have

$$\det \left(\begin{pmatrix} P' & Z \end{pmatrix} \begin{pmatrix} P \\ Z' \end{pmatrix} \right) = \frac{1}{12} T^2 (T^2 - 1)$$

and (78) reduces to²

$$\frac{1}{12} T^2 (T^2 - 1) (\alpha I + Q) = (I + \alpha P'P + (\alpha - 1) ZZ') \quad (79)$$

²Note that the claim has not been proved, but only verified on a computer. The argument need not be based on that claim, though, as it depends only on the uncritical assumption that $\det(P'P + ZZ')$ is some polynomial in T .

Right-hand multiplication by $(I + \alpha P'P)^{-1}$ gives

$$\begin{aligned} \frac{1}{12} T^2 (T^2 - 1) (\alpha I + Q) (I + \alpha P'P)^{-1} &= \left(I + (\alpha - 1) ZZ' (I + \alpha P'P)^{-1} \right) \\ &= \left(I + (\alpha - 1) ZZ' \left(I - \alpha P'P + (\alpha P'P)^2 - (\alpha P'P)^3 + \dots \right) \right) \\ &= (I + (\alpha - 1) ZZ') \end{aligned} \quad (80)$$

By taking determinants on both sides we obtain

$$\frac{\det(\alpha I + Q)}{\det(I + \alpha P'P)} = \frac{12}{T^2 (T^2 - 1)} \det(I + (\alpha - 1) ZZ') \quad (81)$$

The last determinant can be evaluated by means of its Eigenvalues. For any symmetric matrix $A \in \mathfrak{R}^n$, denote the vector of its Eigenvalues by $\Lambda(A)$. Its rank $r(A) \neq n$ gives the number of non-zero Eigenvalues. The vector of these non-zero Eigenvalues is denoted by $\Lambda^+(A) \in \mathfrak{R}^{r(A)}$. The determinant of A is equal to the product of its Eigenvalues.

We have

$$\Lambda(I + (\alpha - 1) ZZ') = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{pmatrix} + (\alpha - 1) \Lambda(ZZ') \quad (82)$$

Further, $r(Z) = 2$ and ZZ' has rank 2 and two non-zero Eigenvalues of unity.

$$\Lambda^+(ZZ') = \Lambda^+(Z'Z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (83)$$

In view of (82) we conclude that $(I + (\alpha - 1) ZZ')$ has $T - 2$ Eigenvalues of one and two Eigenvalues of α . The determinant of $(I + (\alpha - 1) ZZ')$ is the product of its Eigenvalues and we can write

$$\det(I + (\alpha - 1) ZZ') = \alpha^2 \quad (84)$$

Equation (81) reduces thus to

$$\frac{\det(\alpha I + Q)}{\det(I + \alpha P'P)} = \frac{12 \alpha^2}{T^2 (T^2 - 1)} \quad (85)$$

Taking logarithms and re-arranging terms yields

$$\begin{aligned} \log \det (\alpha I + Q) &= \log \det (I + \alpha P' P) + \log 12 \\ &\quad + 2 \log \alpha - 2 \log T - \log (T^2 - 1) \end{aligned} \quad (86)$$

Disregarding constants, the likelihood function (74) can be written as

$$\mathcal{L}(x; \alpha) = -\log \det (I + \alpha P' P) - T \cdot \log R(\alpha) + (T + 2) \log \alpha \quad (87)$$

7. A STATISTICS ESTIMATOR FOR THE VARIANCES

The likelihood estimation described in the preceding section lacks intuitive appeal, and its small-sample properties are difficult to ascertain. As an alternative, another estimator will be devised that is based on the idea that the calculated variances ought to be close to their expectations. The statistics estimator described in this section is derived by equating, at any sample size, the calculated variances with their expectations.

Assume a realization of a trend y (that we can't observe) along with a realization of the time series x (which is taken as a realization of a random variable) for a given set of parameters β , σ_u^2 , and σ_v^2 . According to (46), this gives rise to the estimate \hat{y} as a function of the variance ratio $\alpha = \sigma_u^2 / \sigma_v^2$ and of the time series x which is the sum of trend y and disturbance u :

$$\hat{y} = (I + \alpha P' P)^{-1} (y + u) \quad (88)$$

Since

$$y = (I + \alpha P' P) y - \alpha P' P y \quad (89)$$

and $v = P y$, equation (88) can be written as

$$\hat{y} = y + (I + \alpha P' P)^{-1} (u - \alpha P' v) \quad (90)$$

Pre-multiplication with P gives

$$\hat{v} = v + P (I + \alpha P' P)^{-1} (u - \alpha P' v) \quad (91)$$

In a similar way, from $\hat{u} = x - \hat{y}$ we obtain

$$\hat{u} = u - (I + \alpha P' P)^{-1} (u - \alpha P' v) \quad (92)$$

Thus the estimated errors \hat{u} and \hat{v} are linear functions of the the normal random variables u and v :

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} (I - M) & -\alpha MP' \\ PM & I - \alpha PMP' \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (93)$$

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \left(I - \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} M & M \\ M & M \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \alpha P' \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix} \quad (94)$$

with

$$M := (I + \alpha P'P)^{-1} \quad (95)$$

and their joint distribution can be calculated:

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \mathcal{N} \left(0, \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right) \quad (96)$$

with

$$S_{11} := \sigma_u^2 (I - M)^2 + \sigma_v^2 \alpha^2 MP'PM \quad (97)$$

$$S_{12} := \sigma_u^2 (I - M) MP' - \sigma_v^2 \alpha MP' (I - \alpha PMP') \quad (98)$$

$$S_{21} := \sigma_u^2 PM (I - M) - \sigma_v^2 \alpha (I - \alpha PMP') PM \quad (99)$$

$$S_{22} := \sigma_u^2 PM^2 P' + \sigma_v^2 (I - \alpha PMP')^2 \quad (100)$$

From this, the expectation of the average squared errors can be calculated:

$$E \left\{ \frac{1}{T} \hat{u}' \hat{u} \right\} = \sigma_u^2 \cdot \frac{1}{T} \text{tr} (I - M)^2 + \sigma_u^2 \alpha \frac{1}{T} \text{tr} (M^2 P' P) \quad (101)$$

$$\begin{aligned} E \left\{ \frac{1}{T-1} \hat{v}' \hat{v} \right\} &= \sigma_u^2 \cdot \frac{1}{T-1} \text{tr} (M^2 P' P) \\ &+ \sigma_v^2 \cdot \frac{1}{T-1} \text{tr} (I - \alpha PMP')^2 \end{aligned} \quad (102)$$

Note that

$$\begin{aligned}
 \text{tr}(I - M)^2 + \alpha \text{tr}(M^2 P' P) &= \text{tr}(I - M)^2 + \alpha \text{tr}(M^2 P' P) \\
 &= \text{tr}(I - 2M + (I + \alpha P' P) M^2) \\
 &= \text{tr}(I - M) \\
 &= T - \text{tr}(M)
 \end{aligned} \tag{103}$$

and

$$\begin{aligned}
 \alpha \text{tr}(M^2 P' P) + \text{tr}(I - \alpha P M P')^2 &= \text{tr}(\alpha M^2 P' P + I - 2\alpha M P' P + \alpha^2 M^2 (P' P)^2) \\
 &= \text{tr}(\alpha (I + \alpha P' P) M^2 P' P + I - 2\alpha M P' P) \\
 &= \text{tr}(I - \alpha M P' P)
 \end{aligned} \tag{104}$$

Because

$$M(I + \alpha P' P) = I \tag{105}$$

we have

$$I - \alpha M P' P = M \tag{106}$$

Inserting this into (104) gives

$$\text{tr}(I - M)^2 + \alpha \text{tr}(M^2 P' P) = \text{tr}(M) \tag{107}$$

and (101)-(102) reduce to

$$E \left\{ \hat{u}' \hat{u} \right\} = \sigma_u^2 (T - \text{tr}(M)) \tag{108}$$

$$E \left\{ \hat{v}' \hat{v} \right\} = \sigma_v^2 \text{tr}(M) \tag{109}$$

Our statistics estimators for the variances, denoted by $\check{\sigma}_u^2$ and $\check{\sigma}_v^2$, are obtained by equalizing the estimated moments $\hat{u}' \hat{u}$ and $\hat{v}' \hat{v}$ with their expectations:

$$\hat{u}' \hat{u} = \check{\sigma}_u^2 (T - \text{tr}(M)) \tag{110}$$

$$\hat{v}' \hat{v} = \check{\sigma}_v^2 \text{tr}(M) \tag{111}$$

Note that the estimated moments $\hat{u}'\hat{u}$ and $\hat{v}'\hat{v}$, as implied by (53)-(55) are functions of the observations x and the variance ratio $\check{\alpha} = \check{\sigma}_u^2 / \check{\sigma}_v^2$ and, thus, of the variances $\check{\sigma}_u^2$ and $\check{\sigma}_v^2$ and that the matrix M depends on the variance ratio as well. Hence the solution to (110)-(111) amounts to finding a fixpoint.

The system can be written equivalently as

$$\check{\sigma}_u^2 = \frac{1}{T} \left(\hat{u}'\hat{u} + \alpha \hat{v}'\hat{v} \right) = \frac{1}{T} R(\check{\alpha}) \quad (112)$$

$$\check{\sigma}_v^2 = \frac{\hat{v}'\hat{v}}{\text{tr}(M)} \quad (113)$$

One way of estimating the variances is, thus, to find a fixpoint of (110)-(111) or (112)-(113). Another way is the following.

Consider the function

$$\mathcal{H}(x, \alpha) = -\log \det(I + \alpha P'P) - T \cdot \log R(\alpha) + T \cdot \log \alpha \quad (114)$$

The following Lemma states that the statistics estimator can be derived by maximizing the function $\mathcal{H}(x, \alpha)$.

Lemma 1. *The statistics estimators, as defined by equations (112) and (113), can be obtained by minimizing the function $\mathcal{H}(x, \alpha)$ defined in (114) with respect to α . The variances are computed from the minimizing value $\check{\alpha}$ as*

$$\check{\sigma}_u^2 = \frac{1}{T} R(\check{\alpha}) \quad (115)$$

$$\check{\sigma}_v^2 = \frac{1}{T} \frac{R(\check{\alpha})}{\check{\alpha}} \quad (116)$$

Proof. With $M = (I + \alpha P'P)^{-1}$ we have

$$\begin{aligned} \frac{d \log \det(I + \alpha P'P)}{d\alpha} &= \frac{d}{d\alpha} \left(T \log \alpha + \log \det \left(\frac{1}{\alpha} I + P'P \right) \right) \\ &= \frac{1}{\alpha} (T - \text{tr}(M)) \end{aligned} \quad (117)$$

Note further that

$$\begin{aligned}
 R(\alpha) &= x'(I - M)^2 x + \alpha x' M P' P M x \\
 &= x'(I - 2M + M^2 + \alpha M P' P M) x \\
 &= x'(I - 2M + M(I + \alpha P' P) M) \\
 &= x'(I - M) x \\
 &= x' x - x' M x
 \end{aligned} \tag{118}$$

Consider

$$\begin{aligned}
 \frac{\partial M}{\partial \alpha} &= \frac{\partial (\alpha(\frac{1}{\alpha} I + P' P))^{-1}}{\partial \alpha} \\
 &= -M P' P M
 \end{aligned} \tag{119}$$

From (118) and (119) we obtain

$$\begin{aligned}
 R'(\alpha) &= x' M P' P M x \\
 &= \hat{v}' \hat{v}
 \end{aligned} \tag{120}$$

Using these results, the derivative of $\mathcal{H}(x, \alpha)$ with respect to α is calculated as

$$\frac{\partial \mathcal{H}}{\partial \alpha} = \frac{1}{\alpha} \text{tr}(M) - T \frac{\hat{v}' \hat{v}}{R} \tag{121}$$

Substituting $\hat{v}' \hat{v}$ by $\check{\sigma}_v^2 \text{tr}(M)$ and $\frac{R}{T}$ by $\check{\sigma}_u^2$ gives

$$\frac{\partial \mathcal{H}}{\partial \alpha} = \frac{1}{\alpha} \text{tr}(M) - \frac{\check{\sigma}_v^2 \text{tr}(M)}{\check{\sigma}_u^2} \tag{122}$$

With $\alpha = \frac{\check{\sigma}_u^2}{\check{\sigma}_v^2}$, this reduces to zero. Hence any solution to (112), (113) satisfies the first-order condition for a maximum of \mathcal{H} . Conversely, (115) implies (112), and (113) can be derived by putting (121) to zero and substituting (116) into (121). \square

8. THE RELATIONSHIP BETWEEN THE MAXIMUM-LIKELIHOOD AND THE STATISTICS ESTIMATOR

With the aid of the function \mathcal{H} , the asymptotic equivalence of the statistics estimator and the maximum likelihood estimator can be established easily.

Theorem 3. *The maximum likelihood estimator and the statistics estimator are asymptotically equivalent.*

Proof. We need to concentrate on the estimation of the smoothing parameter α because the formulae that link the variances to α are identical for the two estimators (See (76), (77) and (112), (113)).

Both the likelihood function \mathcal{L} (eq. 87) and the criterion function \mathcal{H} (eq. 114) tend to infinity with increasing T^3 . In order to compare their asymptotic behavior, it is useful to divide both functions by T and to compare the statistics criterion

$$-\frac{1}{T} \log \det (I + \alpha P' P) - \log R(\alpha) + \log \alpha \quad (123)$$

with the likelihood criterion

$$-\frac{1}{T} \log \det (I + \alpha P' P) - \log R(\alpha) + \frac{T+2}{T} \log \alpha \quad (124)$$

As $\frac{T+2}{T} \rightarrow 1$ for $T \rightarrow \infty$, both maximands are asymptotically identical and the theorem is proved. \square

Theorem 3 establishes that the statistics estimator shares the attractive large-sample properties with the likelihood estimator. Conversely, it states that the intuitive interpretation of the statistics estimator - that the calculated variances are equal to their expectations - carries over to the likelihood estimator in an approximate sense. This enhances the intuitive appeal of the maximum likelihood estimator.

9. NOTES ON NUMERICAL PERFORMANCE

A practical example is provided in Figure 1⁴. Reducing the variance ratio by one half or doubling it has no big effect on the qualitative behavior of the trend.

This does not tell much, however, about how well the method recovers the smoothing constant and the variances of the time series. I have done some simulations to obtain an impression about this aspect of performance. It is beyond the scope of the present paper to present a full-fledged Monte-Carlo study, however. The following remarks are intended to convey my overall impression.

³Note that $\log \det (I + \alpha P' P)$ is $O(T)$.

⁴All computations are made using the Mathematica Package by Ludsteck (2004).

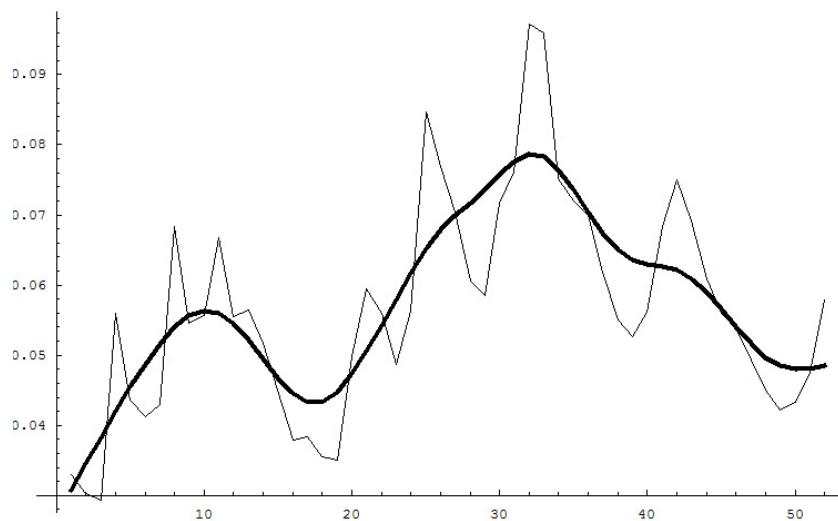


FIGURE 1. US unemployment 1951-2002, source: US Department of Commerce, Bureau of Labor Statistics. Estimated parameters: $\sigma_u^2 = 15$, $\sigma_v^2 = .51$, and $\alpha = 28.8$.

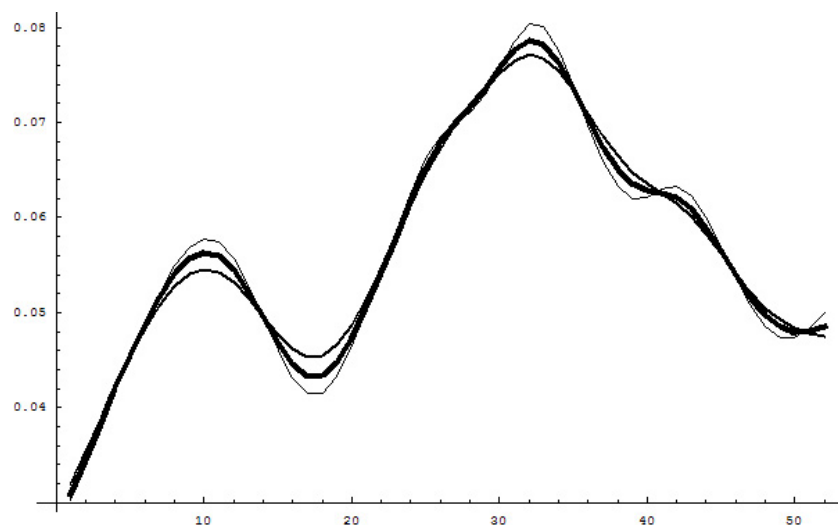


FIGURE 2. Effect of the smoothing constant: $\alpha = 28.8$ (estimated value, heavily drawn curve), $\alpha = 57.6$, and $\alpha = 14.4$ (thin line).

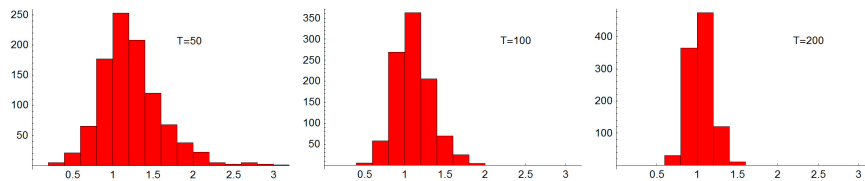


FIGURE 3. Distribution of estimates of $\log_{10}(\alpha)$ for time series of different length with $\sigma_u^2 = 10$ and $\sigma_v^2 = 1$, 1000 trials each.

The critical issue is, of course, how well (and how reliably) the smoothing constant is estimated by the proposed estimator. Given the smoothing constant, the estimation is optimal anyway, and its theoretical distribution is known, see Theorem 1 and equation (50). I concentrate therefore on the estimation of the smoothing constant α , or rather, its logarithm $\log_{10}(\alpha)$, because this seems to be the more relevant quantity.

For the simulations I have done, the method works reasonably well. I just describe a few examples. Figure 3 depicts the frequency distribution for the estimates of the smoothing constant that are obtained by generating 1000 random series according to equations (1), (2), and (7) with variances $\sigma_u^2 = 10$ and $\sigma_v^2 = 1$ (corresponding to a smoothing constant $\log_{10}(\alpha) = 1$) for alternative lengths $T = 50$, $T = 100$, and $T = 200$, respectively.

As expected, the estimates are less reliable for short series and more reliable for long series.⁵ It appears, further, that the median and the mean both tend to overestimate the smoothing constant. With an increasing length of the series, the bias is reduced.

For long time series, the maximum-likelihood estimates and the moments estimates are nearly identical. For short time series, the difference is noticeable. In these examples, the moments estimators work better in the sense of being less biased and pose less numerical difficulties.⁶

⁵The descriptive statistics are as follows:

T	mean	median	min	max	standard deviation
50	1.33	1.18	0.34	14.53	1.18
100	1.11	1.08	0.42	1.91	0.22
200	1.04	1.03	0.67	1.56	0.14

⁶For $T = 20$, the problem is rather ill conditioned. Computations of the moments estimator failed to converge in 6.9% of the cases, as compared to 22% with the maximum-likelihood

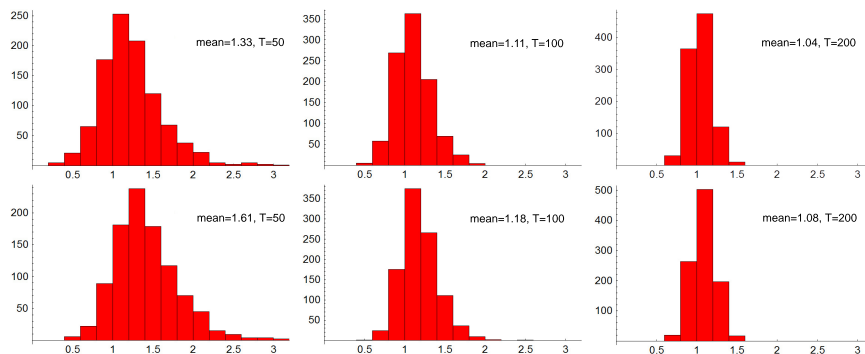


FIGURE 4. Comparison of moments estimators (first row) with maximum-likelihood estimators (second row), $T1 = 100$, 1000 trials each.

The decomposition depends on the smoothing constant, *viz.* the ratio of the variances, rather than the absolute magnitude of the variances (which are affected by scaling). This independence is reconfirmed in the simulations (Figure 5).

estimator. Further, the distribution is two-peaked in both cases. (The figure gives the distribution of the moments estimates (left) and the maximum-likelihood estimates (right)).



No problems of this kind occur with lengths around $T = 40$ and longer, and the peaks at the right-hand side disappear. All this refers to time series with variances of $\sigma_u^2 = 10$ and $\sigma_v^2 = 1$.

Note that I have reason to suspect that the spikes at the right-hand side that occur for small T are computational artefacts produced by a rather flat target function (123). In 200 out of 1000 cases, the number 14.55775742950158 is returned as an estimate for $\log(\alpha)$, for instance, and similar cases of multiplicity sum up to 579 out of 1000 observations. This reduces for longer time series (207 out of 1000 for $T = 25$) and does not occur for still longer time series ($T > 50$). The problem is more pronounced for the maximum-likelihood estimator.

The computations make use of the function `NMaximize[]` of Mathematica 5.0 that implements several advanced numerical methods. Johannes Ludsteck currently works on tuning the method in order to eliminate the results for those ill-conditioned cases that seem responsible for those spikes. If some estimates are reported in following as “truncated”, this refers to sets with eliminated spikes.

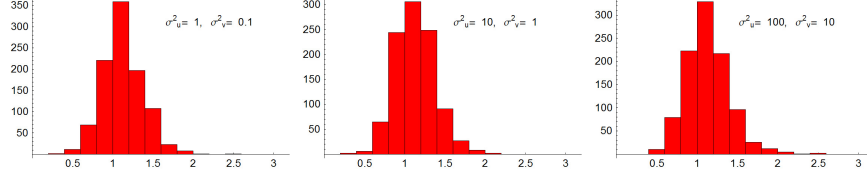


FIGURE 5. Simulations with identical variance ratios but different variances. The means are in the range 1.12 – 1.13, and the standard deviations are in the range 0.24 – 0.27 in all three cases. ($T_2 = 80$ with 1000 trials each.)

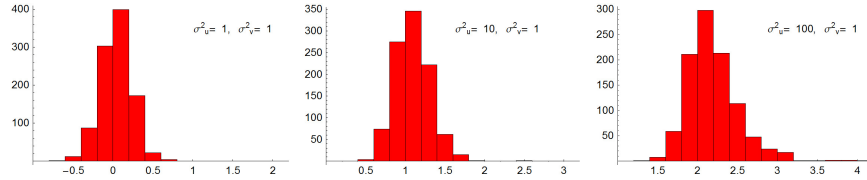


FIGURE 6. Increasing the variance by some factor shifts the distribution of estimates to the right by the same factor. The means are 0.04, 1.09, and 2.19. The standard deviations are 0.19, 0.22, and 0.33. ($T = 100$, 1000 trials each, logarithmic scale.)

Finally, Figure 6 gives the results when variance ratios are changed by a factor of 10. This shifts the distribution on the logarithmic scale to the left or to the right by one unit.

10. NOTES ON MODELLING

The trend filter discussed here has given rise to two strands of thought. One, originally proposed by Schlicht (1984) and also alluded to by Hodrick & Prescott (1997), relates to state-space modelling; the other, starting with King & Rebelo (1993), looks at performance in the frequency domain. The present paper falls into the first category.

The state-space literature tends to rely on Kalman filtering. As Kalman filters are one-sided filters, they are never efficient in the sense of using all available information for estimating trend values at intermediate points in time. The filter proposed in Schlicht (1984) in the context of seasonal adjustment and detailed here for the case of a trend without seasonality is,

in contrast, two-sided and uses all information available. This seems to be a preferable approach.

It seems misleading, however, to use the trend filter discussed here for quarterly or monthly data. This would require integrating the trend filter with a seasonal filter, as in Schlicht (1984), preferably in the Schlicht/Pauly (1983) specification). Otherwise the seasonal pattern would misleadingly be interpreted as uncorrelated noise. For similar reasons, the filter is perhaps not adequate for identifying business cycles. The identification problem for business cycles is similar to the identification of seasonal patterns, but is considerably more involved, as business cycles vary in shape and periodicity.

11. OPEN PROBLEMS

The filter suggest some directions for future research. An obvious open end is that the claim on page 4 which simplifies computation of the estimators considerably, needs to be proved. Another improvement would be to develop a correction for the small-sample bias of the moments estimator. A further problem relates to the modelling and identification of business cycles. A third line of inquiry is suggested by the duality between descriptive and stochastic approaches to seasonal adjustment (Schlicht, 1984). A descriptive dual could enhance our understanding of the way in which the filter processes the raw data in a purely descriptive sense.

REFERENCES

- Hodrick, Robert J., & Prescott, Edward C. 1997. Postwar U.S. Business Cycles: An Empirical Investigation. *Journal of Money, Credit, and Banking*, **29(1)**(February), 1–16.
- King, R. G., & Rebelo, S. T. 1993. Low Frequency Filtering and Real Business Cycles. *Journal of Economic Dynamics and Control*, **17**, 207–31.
- Ludsteck, Johannes. 2004. *HPFilter*. Mathematica Package. Wolfram Research, available at <http://library.wolfram.com/infocenter/MathSource/Mathematics/ProbabilityStatistics/>.
- Schlicht, Ekkehart. 1981. A Seasonal Adjustment Principle and a Seasonal Adjustment Method Derived from This Principle. *Journal of the American Statistical Association*, **76(374)**(June), 374–78. (Paper presented at

- the Econometric Society European Meeting Helsinki 1976, available at <http://www.lrz.de/~ekkehart>).
- Schlicht, Ekkehart. 1984. Seasonal Adjustment in a Stochastic Model. *Statistische Hefte*, **25**, 1–12. (available at <http://www.lrz.de/~ekkehart>).
- Schlicht, Ekkehart. 1985. *Isolation and Aggregation in Economics*. Springer-Verlag Berlin-Heidelberg-New York. (available at <http://epub.ub.uni-muenchen.de/view/subjects/05.html>).
- Schlicht, Ekkehart. 1989. *Variance Estimation in a Random Coefficients Model*. Paper presented at the Econometric Society European Meeting Munich. (available at <http://www.lrz.de/~ekkehart>).
- Schlicht, Ekkehart, & Pauly, Ralf. 1983. Descriptive Seasonal Adjustment by Minimizing Perturbations. *Empirica*. (available at <http://www.lrz.de/~ekkehart>).

UNIVERSITY OF MUNICH, DEPARTMENT OF ECONOMICS, SCHACKSTR. 4, 80539 MUNICH, GERMANY, SCHLICHT@LMU.DE