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## Revenue Maximizing Head Starts in Contests

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## Revenue Maximizing Head Starts in Contests

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in Contests**

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Jörg Franke, Wolfgang Leininger, and Cédric Wasser<sup>1</sup>

# Revenue Maximizing Head Starts in Contests

## Abstract

*We characterize revenue maximizing head starts for all-pay auctions and lottery contests with many heterogeneous players. We show that under optimal head starts all-pay auctions revenue-dominate lottery contests for any degree of heterogeneity among players. Moreover, all-pay auctions with optimal head starts induce higher revenue than any multiplicatively biased all-pay auction or lottery contest. While head starts are more effective than multiplicative biases in all-pay auctions, they are less effective than multiplicative biases in lottery contests.*

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*Keywords: All-pay auction; lottery contest; head start; revenue dominance*

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# 1 Introduction

Competitive situations frequently represent biased contests where specific agents are ex-ante favored in the competitive process through, for instance, affirmative action or outright favoritism by the contest organizer. Often, these forms of asymmetric treatment can be interpreted as granting head starts (or handicaps) to specific agents. Agents that enjoy head starts benefit from their advantageous position in the sense that their rivals must first pass the head start to be able to compete on equal footing. Naturally, the extent of the head starts has strategic implications for favored agents, their respective rivals, and therefore also on the revenue that is generated in the contest. A contest organizer who has the option to fit head starts to the underlying heterogeneity of the contestants can therefore influence the generated contest revenue to some degree.

In this paper we characterize the maximal equilibrium revenue that can be generated in multi-player contest games with the help of appropriately designed head starts. We concentrate on two frameworks that are predominantly used in the contest literature; that is, all-pay auctions with complete information introduced by Hillman and Riley (1989), [7], and lottery contests introduced by Tullock (1980), [16]. Both models are sufficiently tractable and have been extensively applied in various contexts, see Konrad (2009), [10], for a survey. However, our model deviates from the standard setup by allowing the contest organizer to specify idiosyncratic head starts which can be interpreted as granting additive boni to the respective bids or effort levels of favored players. From the perspective of a contest organizer who is interested in maximizing aggregated equilibrium effort or bids (the expected revenue of the contest), specifying the optimal head start becomes then the crucial instrument to increase contest revenue.

Our analysis proceeds by characterizing firstly the respective optimal head starts in the all-pay auction and in the lottery contest framework. More precisely, for the all-pay auction we construct a lower bound on aggregate equilibrium effort based on a specific head start that levels the playing field among those two active contestants who endogeneously decide to participate under the respective head start. For the lottery contest we provide conditions under which positive head starts maximize equilibrium revenue (and derive the corresponding optimal head start for this case). These results allow us to compare the generated contest revenue among the two frameworks which leads to our key result: Under optimal head starts the all-pay auction revenue-dominates the lottery contest for all degrees of heterogeneity. While this result mirrors the revenue ranking for multiplicative biases from Franke et al. (2014), [6], the underlying mechanisms to induce additional revenue with the help of (multiplicative) biases and (additive) head starts are substantially different. In the lottery contest, for instance, the optimal multiplicative bias is specified such that effort exertion by weak contestants is favored which induces additional entry,

comp. Franke et al. (2013), [5]. However, this mechanism does not work with additive head starts because players basically reduce their effort in response to the granted head start. Hence, depending on the underlying heterogeneity either no head start at all is optimal or the head start is specified such that only the strongest player remains active (who then competes as if paired against a fictitious rival whose effort level corresponds to the aggregate head starts of all other non-active players). In the all-pay auction, in contrast, the optimal multiplicative bias is specified such that only the two strongest contestants remain active and the playing field among those is totally balanced, comp. Franke et al. (2014), [6]. Although there is also a leveled playing field in the all-pay auction with optimal head starts, the set of active contestants might not consist of the strongest contestants because depending on the underlying heterogeneity either the two strongest contestants or the strongest and the weakest contestants will be active.

These new revenue comparisons also allow us to establish an unambiguous revenue-ranking with respect to optimal head starts and biases in the two frameworks. The resulting ranking implies, for instance, that revenue in the all-pay auction under optimal head starts is higher than under the optimal bias. Hence, head starts are specifically effective for inducing revenue if the contest is highly competitive as in the all-pay auction. By contrast, multiplicative biases are more effective in the lottery contest as induced revenue in the lottery contest under the optimal bias is higher than under optimal head starts. Nevertheless, the all-pay auction revenue dominates the lottery contest independently of the fact whether the respective optimal head start or the optimal bias is used in the two frameworks. Hence, if revenue maximization is the objective of the contest organizer, then she should resort to the all-pay auction framework with optimal head starts.

Our study complements the mentioned literature on revenue maximization based on asymmetric contest success functions with heterogeneous players. Moreover, our study also contributes to the recent interest in the analysis of head starts in different competitive situations, comp. Kirkegaard (2012), [8], Seel and Wasser (2014), [13], Li and Yu (2012), [11], Segev and Sela (2014), [14], Nti (2004), [12], and Konrad (2002), [9]. However, all of the mentioned studies (as most of the literature) are restricted to the two-player case. In this sense our characterization of the optimal head start in the multi-player all-pay auction framework constitutes, for instance, a direct extension of Li and Yu (2012), [11], to the multi-player case, while a similar relation holds for the lottery contest analyzed in Nti (2004), [12]. To our knowledge there only exist three studies that involve head starts in multi-player all-pay auction or lottery contest frameworks. While Siegel (2014), [15], considers head starts in multi-prize all-pay auctions, Wasser (2013), [17], incorporates homogeneous head starts in a lottery contest framework. However, both studies are rather interested in equilibrium characterization and not in the optimal design of the head



starts. The only paper that explicitly addresses the design of optimal head starts (and biases) in a multi-player framework is Dasgupta and Nti (1998), [4], based on a lottery contest. However, they assume that players and head starts are homogeneous and find that in their specific case no head start at all is optimal. Our study extends this framework by considering heterogeneous contestants and heterogeneous head starts and shows under which conditions these previously established results on optimal head starts still hold under heterogeneity.

The paper is organized as follows. In Section 2 we introduce the model setup and the two frameworks. In Section 3 we determine a lower bound on equilibrium revenue under optimal head starts in the all-pay auction. In Section 4 we analyze under which condition head starts should be used in the lottery contest and identify the corresponding revenue. In Section 5 we compare the induced revenue between the two frameworks and establish a complete revenue-ranking which also incorporates previous result on optimal biases. Section 6 concludes.

## 2 The Model

There are  $n \geq 2$  players of set  $\mathcal{N} = \{1, \dots, n\}$  that compete for an indivisible prize. Players are heterogeneous with respect to their valuations of the prize and can be ordered decreasingly with respect to their valuations:  $v_1 \geq v_2 \geq \dots \geq v_n > 0$ . The probability  $Pr_i(x_i, x_{-i})$  of player  $i \in \mathcal{N}$  to win the prize depends positively on her bid  $x_i \in [0, \infty)$  and negatively on the bids  $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in [0, \infty)^{n-1}$  of her rivals, where all bids are non-refundable. The expected payoff for player  $i \in \mathcal{N}$  is therefore:

$$\pi_i(x_i, x_{-i}) := Pr_i(x_i, x_{-i})v_i - x_i$$

The probability function  $Pr_i(x_i, x_{-i})$  also depends on the specific design of the contest framework. We focus in our analysis on the two most frequently used contest frameworks; that is, a deterministic all-pay auction and a probabilistic lottery contest. More precisely, we consider asymmetric versions of those two contest frameworks where each player  $i \in \mathcal{N}$  potentially benefits from an idiosyncratic head start  $\delta_i \in [0, \infty)$  that is added to player  $i$ 's bid without any costs. Once, the decision on the framework is made and the respective head start  $\delta = (\delta_1, \dots, \delta_n)$  is designed (given the specific framework), the probability for player  $i \in \mathcal{N}$  to win the prize can be expressed as follows:<sup>1</sup>

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<sup>1</sup>Note, that our formulation also includes cases of handicaps (i.e., negative head starts) whenever a positive head start for all other players  $j \neq i$  is equivalent to a negative head start for player  $i$ .

1. For the all-pay auction with head start (framework HAA):

$$Pr_i^{HAA}(x_i, x_{-i}) = \begin{cases} 1, & \text{if } x_i + \delta_i > x_j + \delta_j \text{ for all } j \neq i, \\ \frac{1}{k+1}, & \text{if } x_i + \delta_i = x_j + \delta_j \text{ for } k \text{ agents } j \neq i \text{ and} \\ & x_i + \delta_i > x_l + \delta_l \text{ for all other agents } l \neq i, \\ 0, & \text{if } x_i + \delta_i < x_j + \delta_j \text{ for some } j \neq i. \end{cases}$$

2. For the lottery contest with head start (framework HLC):

$$Pr_i^{HLC}(x_i, x_{-i}) = \begin{cases} \frac{x_i + \delta_i}{\sum_{j=1}^n (x_j + \delta_j)}, & \text{if } \sum_{j=1}^n (x_j + \delta_j) \neq 0, \\ 0, & \text{if } \sum_{j=1}^n (x_j + \delta_j) = 0. \end{cases}$$

In our analysis we are specifically interested in deriving and comparing the maximal contest revenue (the sum of expected equilibrium bids by all players) that can be induced in each framework by specifying optimal head starts. We denote this maximal revenue by  $X^{*,c} := \sum_{i \in \mathcal{N}} E[x_i^{*,c}]$ , where  $E[x_i^{*,c}]$  denotes the expected equilibrium bid of player  $i \in \mathcal{N}$  under the optimal head start in framework  $c \in \{HAA, HLC\}$ . As head starts are framework-specific our analysis proceeds by firstly characterizing the optimal head starts and the corresponding revenue separately in each of the two frameworks and then comparing the maximal revenue between the two frameworks.<sup>2</sup>

For the all-pay auction framework *HAA* we derive a lower bound on maximal revenue based on a specific head start that will depend on the heterogeneity of the players. Our results rely on and extend the two-player analysis of Li and Yu (2012), [11], to the multi-player case. For the lottery contest game we show that positive head starts do not always maximize contest revenue and derive the respective condition on the degree of players' heterogeneity. Given the condition is met we construct the optimal head start which enables us to determine maximal revenue in the lottery contest. The obtained results are then used to prove the revenue-dominance of the all-pay auction with optimal head starts over the lottery contest with optimal head starts for any degree of heterogeneity. Moreover, a comparison with previous results for biased contest games from Franke et al. (2014), [6], implies that the all-pay auction with optimal head starts also revenue dominates unambiguously any multiplicatively biased all-pay auction or lottery contest.

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<sup>2</sup>Our analysis can be alternatively framed as the design problem of a contest organizer, whose objective function depends positively on contest revenue, and whose strategy space consists of the type of contest framework with type-dependent head starts.

### 3 Maximal Revenue in the All-Pay Auction

For the two player case the all-pay auction with head starts has been intensively analyzed, for instance, in Li and Yu (2012), [11]. Our analysis of the multi-player all-pay auction is based on these results in the following way: We first restrict the set of feasible head starts to those under which exactly two players from set  $\mathcal{N}$  are active. This allows us to obtain closed form expressions for total revenue given this specific class of head starts. We then identify the optimal pair of active players and the corresponding head start which implies that depending on the underlying heterogeneity either the two strongest players or the strongest and the weakest player will decide to be active. The resulting revenue from this head start must then constitute a lower bound for maximal revenue under optimal head starts chosen from the non-restricted set. We state this lower bound in the subsequent proposition.<sup>3</sup>

**Proposition 3.1** *An optimal head start  $\delta^*$  in the HAA framework yields total equilibrium revenue that satisfies*

$$X^{*,HAA} \geq \max \left\{ \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}, \frac{v_n^2}{2v_1} + v_1 - \frac{v_n}{2} \right\}.$$

**Proof.** Consider the head start  $\delta_{i,j} = (\delta_1, \dots, \delta_n)$ , where  $\delta_i = \delta_j = v_1$  with  $i < j$  and  $\delta_k = 0$  for all  $k \neq i, j$ . Under this specific head start player  $i$  and  $j$  will be active and bid positive amounts while all other players will remain inactive (as they will never bid more than  $v_1$ ). Now apply Li and Yu (2012), [11], Proposition 2, to construct an optimal head start that exactly neutralizes the difference in valuation between the two active players and leaves all other players still inactive: Head start  $\delta_{i,j}^* = (\delta_1^*, \dots, \delta_n^*)$  with  $\delta_i^* = \delta_i$ ,  $\delta_j^* = \delta_j + (v_i - v_j)$ , and  $\delta_k^* = \delta_k = 0$  for all  $k \neq i, j$ , satisfies this requirement because  $\delta_j^* - \delta_i^* = v_i - v_j \geq 0$  such that the playing field is leveled between player  $i$  and  $j$  without affecting non-active contestants. Expected revenue is then:  $X^{HAA}(\delta_{i,j}^*) = \frac{v_j^2}{2v_1} + v_i - \frac{v_j}{2}$ .

Note that  $\frac{\partial X^{HAA}(\delta_{i,j}^*)}{\partial v_i} = 1 - \frac{1}{2} \left( \frac{v_j}{v_1} \right)^2 > 0$ , which implies that the head start should be specified such that  $i = 1$ . Expected revenue is then  $X^{HAA}(\delta_{1,j}^*) = \frac{v_j^2}{2v_1} + v_1 - \frac{v_j}{2}$ . Note that  $\frac{\partial X^{HAA}(\delta_{1,j}^*)}{\partial v_j} = \frac{v_j}{v_1} - \frac{1}{2}$ . Hence,  $X^{HAA}(\delta_{1,j}^*)$  is increasing for relatively high values of  $v_j$  and decreasing for low values of  $v_j$ . This implies that the bias must be specified such that either  $j = 2$  or  $j = n$ . The first case revenue dominates the second if  $X^{HAA}(\delta_{1,2}^*) > X^{HAA}(\delta_{1,n}^*) \Leftrightarrow \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} > \frac{v_n^2}{2v_1} + v_1 - \frac{v_n}{2} \Leftrightarrow v_1 - v_2 < v_n$ .

<sup>3</sup>An alternative approach would be to construct the mixed-strategy equilibria in explicit form for the multi-player all-pay auction with head starts. However, this is challenging because there presumably exists a continuum of equilibria that are potentially not even revenue-equivalent (similar to the equilibrium characterization of the symmetric all-pay auction in Baye et al. (1996), [2]). It should also be mentioned that our setup can neither be transformed into a symmetric all-pay auction, nor is it possible to apply Siegel (2014), [15], to (a transformed version of) our setup because his genericity assumption would preclude the use of the optimal head start  $\delta^*$ .

Hence, if  $v_1 - v_2 < v_n$  then the head start should be specified such that only player 1 and 2 are active and  $\delta_2^* - \delta_1^* = v_1 - v_2 \geq 0$  which yields expected revenue of  $\frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}$ . If  $v_1 - v_2 > v_n$  then the head start should be specified such that player 1 and  $n$  are active with  $\delta_n^* - \delta_1^* = v_1 - v_n \geq 0$ , which results in expected revenue of  $\frac{v_n^2}{2v_1} + v_1 - \frac{v_n}{2}$ .

Allowing any feasible head start (where potentially more than two players are active in equilibrium) must therefore result in maximal revenue of at least  $\max \left\{ \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}, \frac{v_n^2}{2v_1} + v_1 - \frac{v_n}{2} \right\}$ .  $\square$

Note that the inequality in Proposition 3.1 holds as equality for  $n = 2$  (as shown by Li and Yu (2012), [11]). Moreover, one can show that a 2-player contest with head start  $\delta = (\delta_1, \delta_2)$ , where  $0 \leq \delta_2 - \delta_1 \leq v_1 - v_2$  such that the advantage in head starts for the weaker player 2 is not larger than his disadvantage with respect to valuations,<sup>4</sup> yields the following equilibrium payoffs and revenue:

$$\begin{aligned} \pi_1(\delta_1, \delta_2) &= (v_1 - v_2) - (\delta_2 - \delta_1) \\ \pi_2(\delta_1, \delta_2) &= 0 \\ X^{HAA}(\delta_1, \delta_2) &= \frac{1}{2}v_2 + \frac{v_2^2}{2v_1} + (\delta_2 - \delta_1) \end{aligned}$$

Hence, in equilibrium a head start advantage of  $\delta_2 - \delta_1 > 0$  for the weaker contestant precisely transfers  $(\delta_2 - \delta_1)$  units of payoff from the stronger player 1 to the contest revenue. The result of Li and Yu's (2012), [11], then says that a revenue maximizing contest organizer can direct all of the expected profit of  $(v_1 - v_2)$ , that player 1 obtains in the equilibrium with zero head starts, away from player 1 into his own pocket by applying the optimal head start  $\delta_2^* - \delta_1^* = v_1 - v_2$  which then yields equilibrium revenue of  $X^{HAA}(\delta_1^*, \delta_2^*) = \frac{1}{2}v_2 + \frac{v_2^2}{2v_1} + (v_1 - v_2) = \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}$ .

This should be contrasted with the situation under an optimal multiplicative bias, compare Franke et al. (2014), [6]. Also there, the optimal bias implies that the equilibrium payoff of the strongest contestant is reduced to zero, however, this payoff reduction of  $v_1 - v_2$  is not entirely transformed into revenue. In fact, in Franke et al. (2014), [6], it is shown that there is some loss involved in this transformation because revenue is only increased by  $\frac{v_1 + v_2}{2v_1}(v_1 - v_2)$ . As  $\frac{v_1 + v_2}{2v_1} < 1$  the increase in revenue due to the optimal bias is lower than under the optimal head start.

<sup>4</sup>It suffices to consider this case for maximal revenue considerations as it cannot be optimal for the contest organizer to apply head starts that turn the weaker player 2 into the effectively stronger player by setting  $\delta_2 - \delta_1 > v_1 - v_2$ .

## 4 Optimal Head Starts in Lottery Contests

Our analysis of the lottery contest framework starts with an explicit characterization of the equilibrium given a specific head start  $\delta$  by applying the method of Wasser (2013), [17]. We then show that a positive head start (where  $\delta_i > 0$  for at least one player) can never be optimal if it results in at least two players being active. Based on this result we then analyze the remaining case with one active player and derive a condition under which a positive head start is optimal. We construct the respective revenue maximizing head start in this case and determine the corresponding maximal revenue.

The equilibrium characterization is based on the share function approach, suggested by Cornes and Hartley (2005), [3]. We therefore perform a change of variables  $y_i := x_i + \delta_i$  for all  $i \in \mathcal{N}$ , denote  $Y := \sum_{j=1}^n y_j$ , and normalize payoffs with respect to individual valuations  $v_1, \dots, v_n$ . Consider then the equivalent game where each player  $i \in \mathcal{N}$  chooses  $y_i \in [\delta_i, \infty)$  and obtains payoff

$$\hat{\pi}_i(y_1, \dots, y_n) = \frac{y_i}{Y} - \frac{y_i - \delta_i}{v_i}.$$

The first order condition for  $i$ 's effort choice is

$$\frac{Y - y_i}{Y^2} - \frac{1}{v_i} \leq 0, \quad \text{with equality if } y_i > \delta_i.$$

From this we obtain  $y_i = \max\left\{Y - \frac{Y^2}{v_i}, \delta_i\right\}$  and the share function

$$s_i(Y) = \frac{y_i}{Y} = \max\left\{1 - \frac{Y}{v_i}, \frac{\delta_i}{Y}\right\}.$$

Define  $f_i(Y) := 1 - \frac{Y}{v_i}$  and  $g_i(Y) := \frac{\delta_i}{Y}$  such that  $s_i(Y) = \max\{f_i(Y), g_i(Y)\}$ . Let the aggregate share function be denoted by  $S(Y) := \sum_{i=1}^n s_i(Y)$ .

An effort profile  $x_1, \dots, x_n$  in the original contest is a pure-strategy Nash equilibrium if and only if there is a  $Y^*$  such that  $x_i = s_i(Y^*)Y^* - \delta_i$  for each  $i \in \mathcal{N}$  and  $S(Y^*) = 1$ . Note that for players  $i$  who are active in equilibrium ( $x_i > 0$ ) we have  $s_i(Y^*) = f_i(Y^*)$ , whereas for inactive players ( $x_i = 0$ ) we have  $s_i(Y^*) = g_i(Y^*)$ .

It is straightforward to show that there is a unique pure-strategy Nash equilibrium.  $S(Y)$  is continuous and strictly decreasing. Moreover,  $S\left(\sum_j \delta_j\right) \geq \sum_i g_i\left(\sum_j \delta_j\right) = 1$  and  $S(Y) < 1$  for  $Y$  large enough. Hence, there is a unique  $Y$  that solves  $S(Y) = 1$ .

Suppose there is an equilibrium where the nonempty subset of players  $\mathcal{K} \subseteq \mathcal{N}$  is active,

whereas the remaining players are inactive. Hence,  $S(Y) = 1$  is equivalent to

$$S(Y) = \sum_{i \in \mathcal{K}} f_i(Y) + \sum_{j \notin \mathcal{K}} g_j(Y) = |\mathcal{K}| - Y \sum_{i \in \mathcal{K}} 1/v_i + \frac{1}{Y} \sum_{j \notin \mathcal{K}} \delta_j = 1.$$

Solving this equation for  $Y$  we obtain

$$Y = Y(\mathcal{K}) := \frac{|\mathcal{K}| - 1 + \sqrt{(|\mathcal{K}| - 1)^2 + 4(\sum_{i \in \mathcal{K}} 1/v_i)(\sum_{j \notin \mathcal{K}} \delta_j)}}{2 \sum_{i \in \mathcal{K}} 1/v_i}.$$

In addition, define  $Y(\emptyset) := \sum_i \delta_i$ . Hence, we have now defined a function  $Y(\mathcal{K})$  for all  $2^n$  possible subsets  $\mathcal{K}$  of the set  $\mathcal{N}$  of players.

We will now show that the function  $Y(\mathcal{K})$  is maximized at the subset of players  $\mathcal{K}$  that are indeed active in equilibrium. Consider a set of active players  $\mathcal{M}$  that does not correspond to the equilibrium. By definition, we have

$$\sum_{i \in \mathcal{M}} f_i(Y(\mathcal{M})) + \sum_{j \notin \mathcal{M}} g_j(Y(\mathcal{M})) = 1.$$

However, since  $\mathcal{M}$  does not correspond to an equilibrium, we must have at least one  $i \in \mathcal{M}$  where  $f_i(Y(\mathcal{M})) < g_i(Y(\mathcal{M}))$  or one  $j \notin \mathcal{M}$  where  $g_j(Y(\mathcal{M})) < f_j(Y(\mathcal{M}))$ . Hence,  $S(Y(\mathcal{M})) > 1$ . Since  $S$  is strictly decreasing, we must have  $Y(\mathcal{M}) < Y(\mathcal{K})$ , where  $\mathcal{K}$  is the unique subset of players that satisfies  $S(Y(\mathcal{K})) = 1$ .

The following proposition summarizes our findings.

**Proposition 4.1** *There is a unique pure-strategy Nash equilibrium for any valuation profile  $(v_1, \dots, v_n)$  in the HLC framework. Let*

$$\mathcal{K}^* = \arg \max_{\mathcal{K} \subseteq \{1, \dots, n\}} Y(\mathcal{K}).$$

*In equilibrium, players  $i \in \mathcal{K}^*$  exert effort*

$$x_i = Y(\mathcal{K}^*) - \frac{Y(\mathcal{K}^*)^2}{v_i} - \delta_i > 0, \quad (1)$$

*whereas players  $j \notin \mathcal{K}^*$  exert zero effort.*

Moreover, note that  $f_i(Y) \leq g_i(Y)$  for all  $Y$  if  $v_i \leq 4\delta_i$ .<sup>5</sup> Hence,  $v_i \leq 4\delta_i$  is a sufficient condition

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<sup>5</sup>  $f_i(Y) \leq g_i(Y)$  is equivalent to  $Y - \frac{Y^2}{v_i} - \delta_i \leq 0$ . The LHS is maximized at  $Y = \frac{v_i}{2}$ . Hence, the inequality holds if

for player  $i$  to be inactive, which is independent of the cost and head start parameters of the other players.

Based on this equilibrium characterization the expression for induced revenue in the lottery contest for a given head start  $\delta$  can be simplified as follows:<sup>6</sup>

$$X^{HLC}(\delta) := \sum_{i \in \mathcal{N}} x_i(\delta) = \sum_{i \in \mathcal{K}^*(\delta)} x_i(\delta) = Y(\delta, \mathcal{K}^*(\delta)) - \sum_{i=1}^n \delta_i.$$

Hence, the maximal revenue in the lottery contest can be obtained as the solution to the maximization problem  $\max_{\delta \in [0, \infty)^n} Y(\delta, \mathcal{K}^*(\delta)) - \sum_{i=1}^n \delta_i$ , where  $\mathcal{K}^*(\delta) = \arg \max_{\mathcal{K} \subseteq \{1, \dots, n\}} Y(\delta, \mathcal{K})$ .

We solve this maximization problem by analyzing two cases: In the first case we consider head starts under which at least two players are active and show that applying no head starts instead, i.e., applying the zero head start  $\delta_0 := (0, \dots, 0)$ , induces higher revenue. In the second case we consider head starts under which exactly one player is active, derive the revenue maximizing head start and the respective revenue, and compare the results with those from the first case.

**Proposition 4.2** *A lottery contest with zero head start  $\delta_0$  induces higher equilibrium revenue than a lottery contest with positive head starts  $\delta$  provided that  $|\mathcal{K}^*(\delta)| \geq 2$  players are active.*

**Proof.** Observe that, holding  $\mathcal{K}$  fixed, we have

$$\begin{aligned} \frac{\partial Y(\delta, \mathcal{K})}{\partial \delta_i} &= 0 \quad \text{for } i \in \mathcal{K}, \\ \frac{\partial Y(\delta, \mathcal{K})}{\partial \delta_j} &= \frac{1}{\sqrt{(|\mathcal{K}| - 1)^2 + 4(\sum_{i \in \mathcal{K}} 1/v_i)(\sum_{j \notin \mathcal{K}} \delta_j)}} \leq \frac{1}{|\mathcal{K}| - 1} \quad \text{for } j \notin \mathcal{K}, \end{aligned} \quad (2)$$

such that  $\frac{\partial Y(\delta, \mathcal{K}) - \sum_i \delta_i}{\partial \delta_j} \leq 0$  for all  $j \in \mathcal{N}$  whenever  $|\mathcal{K}| \geq 2$ .

Suppose  $\delta$  is such that  $|\mathcal{K}^*(\delta)| \geq 2$ . This leads to the following chain of (in-)equalities which proves the result:

$$X^{HLC}(\delta) = Y(\delta, \mathcal{K}^*(\delta)) - \sum_{i=1}^n \delta_i < Y(\delta_0, \mathcal{K}^*(\delta)) \leq Y(\delta_0, \mathcal{K}^*(\delta_0)) = X^{HLC}(\delta_0).$$

□

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<sup>6</sup> $\frac{v_i}{2} - \frac{1}{v_i} \frac{v_i^2}{4} - \delta_i \leq 0$ , which can be simplified to  $v_i \leq 4\delta_i$ .

<sup>6</sup>From now on we change notation slightly to clarify that equilibrium values depend on the head start  $\delta$ .

Before we proceed to the second case where head starts involve only one active player, the previous result should be put in context with respect to multiplicatively biased contest games, see Franke et al. (2013), [5]. In fact, Proposition 4.2 is in stark contrast to the characterization of the optimal (multiplicative) bias in lottery contests where the optimal bias is used to induce additional entry of contestants to increase revenue. In the lottery contest with head starts, however, Proposition 4.2 implies that any positive head start under which at least two players are active can only yield less revenue. Hence, head starts can never be used to increase revenue by inducing additional entry, which would result in a higher number of active players than in the case without head starts (where at least two players are active). Therefore, the only way left to increase contest revenue beyond the zero head start level is to decrease the number of active players to one by designing suitable head starts. This sounds a priori unlikely to work; however, it should be noted that the remaining single active player has to compete against the sum of all head starts of the other, inactive players. Increasing this sum by increasing the head start of at least one inactive player leaves all of these players inactive while the remaining active player has to counter a higher total of others' head starts with possibly higher effort of his own. Hence, contest revenue may increase if this argument does not alter the participation decisions of at least one player. The following result provides a precise condition on the heterogeneity of valuations under which an optimal head start involving only one active player yields higher contest revenue than a contest without head starts.

For notational convenience we use the following notation for the set of active players in lottery contests without head starts (i.e., where the zero head start  $\delta_0$  is applied):  $\mathcal{K}_0 = \mathcal{K}^*(\delta_0)$  and  $k_0 = |\mathcal{K}^*(\delta_0)|$ .

**Proposition 4.3** *If  $\frac{v_1}{4} > \frac{k_0-1}{\sum_{i \in \mathcal{K}_0} 1/v_i}$ , then there exist positive head starts  $\delta^*$  in the HLC framework that result in equilibrium revenue  $X^{HLC}(\delta^*) = \frac{v_1}{4} > X^{HLC}(\delta_0)$ . Otherwise,  $X^{HLC}(\delta_0) \geq X^{HLC}(\delta)$  for any  $\delta \in [0, \infty)^n$ .*

The proof proceeds with the help of three lemmata. In the first lemma revenue maximizing head starts  $\delta^*$  are characterized under the assumption that only one player is active. In the second lemma a condition is derived such that a contest game with head start  $\delta^*$  induces higher aggregated effort than a contest game with zero head starts. In the third lemma it is shown that under the previously identified condition it is possible to construct a  $\delta^*$  such that in equilibrium only player 1 is active. The one-player solution is optimal, whenever heterogeneity among active players is sufficiently high; for instance, if  $v_{k_0} < \frac{v_1}{4}$ , then the first assumption in Proposition 4.3 always holds.



**Lemma 4.4** *Under the assumption that only one player is active, optimal head starts are of the following form:*

$$\delta^* := (\delta_1^*, \dots, \delta_n^*), \text{ where } \delta_1^* = 0 \text{ and } \sum_{j=2}^{k_0} \delta_j^* = \frac{v_1}{4}. \quad (3)$$

**Proof.** Assume that under head start  $\delta$  player  $i$  is the only active player. Let  $D_i := \sum_{j \neq i} \delta_j$  be the sum of head starts of player  $i$ 's rivals. The expected payoff function of player  $i$  under this head start becomes  $u(x_i, 0, \dots, 0) = \frac{x_i + \delta_i}{x_i + \delta_i + D_i} v_i - x_i$ , which is maximized for  $x_i(\delta) := \sqrt{D_i v_i} - \delta_i - D_i$ . As player  $i$  is the only active player, equilibrium revenue is equal to individual effort:  $X^{HLC}(\delta) = x_i(\delta)$ . Maximizing this function with respect to  $\delta$  implies that an optimal head start  $\delta^*$  should satisfy  $\delta_i^* = 0$  and  $D_i^* = \frac{v_i}{4}$  which results in contest revenue of  $X^{HLC}(\delta^*) = \frac{v_i}{4}$ . This function is increasing in  $v_i$ . Hence, the optimal head start should be specified such that player 1 is the active player which leads to the optimal head start specification in (3).  $\square$

Note, that the statement in Lemma 4.4 effectively means that the zero head start solution among two equally strong contestants with  $v_1 = v_2$  is copied; that is, the strongest contestant 1 is made to compete as if he would face an equally strong opponent. This amounts to a 'levelling the field'-result.

**Lemma 4.5** *Assume that  $\frac{v_1}{4} > \frac{k_0 - 1}{\sum_{i \in \mathcal{K}_0} 1/v_i}$  and that only one player is active under a head start  $\delta^*$  satisfying eq. (3). Then,  $X^{HLC}(\delta^*) > X^{HLC}(\delta_0)$ . Otherwise,  $X^{HLC}(\delta^*) \leq X^{HLC}(\delta_0)$ .*

**Proof.** In a contest game without head start aggregated effort is  $X^{HLC}(\delta_0) = \frac{k_0 - 1}{\sum_{i \in \mathcal{K}_0} 1/v_i}$ , where  $k_0$  is the largest integer that satisfies  $\frac{k_0 - 1}{v_{k_0}} < \sum_{i \in \mathcal{K}_0} 1/v_i$ . The inequality  $X^{HLC}(\delta^*) > X^{HLC}(\delta_0)$  leads to the condition in the lemma.  $\square$

**Lemma 4.6** *Given that  $\frac{v_1}{4} > \frac{k_0 - 1}{\sum_{i \in \mathcal{K}_0} 1/v_i}$  there exist optimal head starts satisfying eq. (3) such that only player 1 is active in equilibrium.*

**Proof.** The proof proceeds in two steps. First, we start from a contest game without any head start and construct a head start  $\hat{\delta}$  such that only player 1 is active in equilibrium. We then modify  $\hat{\delta}$  to get to the optimal head start  $\delta^*$  without affecting the set of active or non-active players.

1. Step: Without head starts all players  $i \in \{1, \dots, k_0\}$  are active and exert  $x_i(\delta_0) > 0$  in equilibrium, while all players  $j \in \{k_0 + 1, \dots, n\}$  are non-active and exert  $x_j(\delta_0) = 0$ .

Note that for any positive head start granted to active players, eq. (1) and (2) imply that  $\frac{\partial x_i(\delta)}{\partial \delta_i} = -1$  and  $\frac{\partial x_i(\delta)}{\partial \delta_j} = 0$  for all active players  $i, j \in \{1, \dots, k_0\}$ . Hence, active players that are directly affected by this type of head start reduce their equilibrium effort exactly by the amount of their individual head start. This implies that  $x_i(\delta) = x_i(\delta_0) - \delta_i$  such that 'effective' effort  $y_i(\delta) = x_i(\delta) + \delta_i$  remains constant under any head start  $\delta_i \in [0, x_i(\delta_0)]$  granted to active players from set  $\mathcal{K}_0$  because  $y_i(\delta) = x_i(\delta) + \delta_i = x_i(\delta_0) = y(\delta_0)$ .

Consider now a head start  $\hat{\delta}$  with  $\hat{\delta}_1 = 0$ ,  $\hat{\delta}_i = x_i(\delta_0)$  for all  $i \in \{2, \dots, k_0\}$  and  $\hat{\delta}_j = 0$  for all  $j \notin \mathcal{K}_0$ . By the previous argument  $y_i(\hat{\delta}) = x_i(\hat{\delta}) + \hat{\delta}_i = 0 + \hat{\delta}_i = x_i(\delta_0) = y_i(\delta_0)$  for all players  $i \in \{2, \dots, k_0\}$ , which also implies that the strategic situation for all other players is the same as before. The payoff function of player 1, for instance, who is the only player that remains active, can then be expressed as  $u(x_1, 0, \dots, 0) = \frac{x_1}{x_1 + \hat{D}_1} v_1 - x_1$ , where  $\hat{D}_1 = \sum_{i=2}^{k_0} \hat{\delta}_i$ . Note, that this is exactly the same payoff as in the case with zero head starts:  $u(x_1, x_2(\delta_0), \dots, x_n(\delta_0)) = \frac{x_1}{x_1 + \sum_{j=2}^n x_j(\delta_0)} v_1 - x_1$ . Hence,  $x_i(\hat{\delta}) = x_i(\delta_0)$  for all players  $i \in N$ . Note also, that  $\hat{D}_1 < \sum_{j \in \mathcal{K}_0} x_j(\delta_0) < \frac{v_1}{4}$ .

2. Step: Starting from  $\hat{\delta}$  the head start can be further modified by increasing individual head starts  $(\hat{\delta}_2, \dots, \hat{\delta}_n)$  until  $D_1^* = \frac{v_1}{4}$  is obtained. This modification will not alter the set of active (player 1) or non-active players (however, the effort level of player 1 will be affected positively). Hence, player 1 will be the only active player and the resulting modified head start corresponds to  $\delta^*$  as characterized in eq. (3).

□

Proposition 4.3 implies that the optimal head start in lottery contests is either  $\delta_0$  (zero head starts for all players) or  $\delta^*$  depending on the underlying heterogeneity. This allows us to determine the maximal revenue that can be induced in a lottery contest with head starts.

**Proposition 4.7** *An optimal head start in the HLC framework yields equilibrium revenue*

$$X^{*,HLC} = \max \left\{ \frac{v_1}{4}, \frac{k_0 - 1}{\sum_{i \in \mathcal{K}_0} 1/v_i} \right\}, \text{ where } \mathcal{K}_0 = \left\{ j \in N \mid \frac{j-1}{v_j} < \sum_{i=1}^j 1/v_i \right\}.$$

## 5 Revenue Ranking

Based on the previous results we are now in a position to compare maximal revenue under optimal head starts in the all-pay auction and the lottery contest. The following proposition estab-

lishes revenue-dominance of the all-pay auction with optimal head starts over the lottery contest with optimal head starts for any degree of heterogeneity.

**Proposition 5.1** *The HAA framework induces higher equilibrium revenue than the HLC framework:  $X^{*,HAA} > X^{*,HLC}$ . This holds for any given set of valuations  $v_1 \geq v_2 \geq \dots \geq v_n \geq 0$ .*

**Proof.** Based on Proposition 3.1 and 4.7 it is sufficient to show that the following inequality is satisfied:

$$\max \left\{ \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}, \frac{v_n^2}{2v_1} + v_1 - \frac{v_n}{2} \right\} > \max \left\{ \frac{v_1}{4}, \frac{k_0 - 1}{\sum_{i \in K_0} 1/v_i} \right\}. \quad (4)$$

We consider the following two cases:

- (i) Suppose that  $\max \left\{ \frac{v_1}{4}, \frac{k_0 - 1}{\sum_{i \in K_0} 1/v_i} \right\} = \frac{v_1}{4}$ . Then the inequality in (4) holds if  $\frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} > \frac{v_1}{4}$  because  $\max \left\{ \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}, \frac{v_n^2}{2v_1} + v_1 - \frac{v_n}{2} \right\} \geq \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}$ . Note, that  $\frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} > \frac{v_1}{4}$  can be reduced to  $(v_2 - v_1)^2 + v_1 v_2 + \frac{1}{2} v_1^2 > 0$  which always holds.
- (ii) Suppose that  $\max \left\{ \frac{v_1}{4}, \frac{k_0 - 1}{\sum_{i \in K_0} 1/v_i} \right\} = \frac{k_0 - 1}{\sum_{i \in K_0} 1/v_i}$ . From Theorem 4.3 in Franke et al. (2014), [6], we know that  $\frac{v_1 + v_2}{2} > \frac{k_0 - 1}{\sum_{i \in K_0} 1/v_i}$ , because the optimally biased all-pay auction strictly revenue-dominates the optimally biased lottery contest which by itself induces higher revenue than the unbiased lottery which is equivalent to a lottery contest with zero head starts. Now observe that  $\frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} \geq \frac{v_1 + v_2}{2}$  because this inequality can be reduced to  $(v_1 - v_2)^2 \geq 0$  which always holds. Hence, we established the following chain of inequalities:  $\max \left\{ \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}, \frac{v_n^2}{2v_1} + v_1 - \frac{v_n}{2} \right\} \geq \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} \geq \frac{v_1 + v_2}{2} > \frac{k_0 - 1}{\sum_{i \in K_0} 1/v_i}$ .  $\square$

Hence, the revenue-ordering between the all-pay auction and the lottery contest with optimal head starts is the same as between the all-pay auction and the lottery contest with optimal multiplicative biases (frameworks BAA and BLC), see Franke et al. (2014), [6]. Combining the two revenue-dominance results yields a complete ranking of all four frameworks independent of the underlying heterogeneity.

**Proposition 5.2** *There is an unambiguous revenue-ranking of all-pay auctions and lottery contests with optimal head starts or multiplicative biases, respectively, for any given set of valuations  $v_1 \geq v_2 \geq \dots \geq v_n \geq 0$ :*

$$X^{*,HAA} \geq X^{*,BAA} > X^{*,BLC} \geq X^{*,HLC}. \quad (5)$$

Moreover, if  $v_1 > v_2$  then the revenue-ranking is strict:

$$X^{*,HAA} > X^{*,BAA} > X^{*,BLC} > X^{*,HLC}.$$

**Proof.** The first inequality of (5) holds because  $X^{*,HAA} \geq \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} \geq \frac{v_1+v_2}{2} = X^{*,HAA}$ , see the proof of case (ii) in Proposition 5.1. Note also, that  $v_1 > v_2$  implies that  $X^{*,HAA} \geq \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} > \frac{v_1+v_2}{2} = X^{*,HAA}$ . The second inequality of (5) holds because the optimally biased all-pay auction revenue-dominates the optimally biased lottery contest, see Theorem 4.3 in Franke et al. (2014), [6]. It remains to prove the third inequality of (5) which can be stated as follows:

$$X^{*,BLC} = \frac{1}{4} \left[ \sum_{i \in \hat{\mathcal{K}}} v_i - \frac{(\hat{k} - 2)^2}{\sum_{j \in \hat{\mathcal{K}}} 1/v_j} \right] \geq X^{*,HLC} = \max \left\{ \frac{v_1}{4}, \frac{k_0 - 1}{\sum_{i \in \mathcal{K}_0} 1/v_i} \right\}, \quad (6)$$

where  $\hat{k}$  satisfies  $\frac{\hat{k}-2}{v_{\hat{k}}} < \sum_{i \in \hat{\mathcal{K}}} 1/v_i$ , see Proposition 4.1 in Franke et al. (2014), [6] (or the original derivation in Franke et al. (2013), [5]), and  $k_0$  satisfies  $\frac{k_0-1}{v_{k_0}} < \sum_{i \in \mathcal{K}_0} 1/v_i$ , see the proof of Lemma 4.5. We consider the following two cases:

- (i) Suppose that  $\max \left\{ \frac{v_1}{4}, \frac{k_0-1}{\sum_{i \in \mathcal{K}_0} 1/v_i} \right\} = \frac{v_1}{4}$ . Then the inequality in (6) holds if  $\sum_{i \in \hat{\mathcal{K}}} v_i - \frac{(\hat{k}-2)^2}{\sum_{j \in \hat{\mathcal{K}}} 1/v_j} > v_1$ . Note, that  $\sum_{i \in \hat{\mathcal{K}}} v_i - \frac{(\hat{k}-2)^2}{\sum_{j \in \hat{\mathcal{K}}} 1/v_j} > \sum_{i \in \hat{\mathcal{K}}} v_i - (\hat{k} - 2)v_{\hat{k}}$  because  $\frac{\hat{k}-2}{v_{\hat{k}}} < \sum_{i \in \hat{\mathcal{K}}} 1/v_i$ . Hence, the previous inequality holds if  $\sum_{i \in \hat{\mathcal{K}}} v_i - (\hat{k} - 2)v_{\hat{k}} > v_1$ . This inequality can be reformulated as  $v_1 + v_2 + \sum_{i=3}^{\hat{k}} (v_i - v_{\hat{k}}) > v_1$  which always holds because  $v_i \geq v_{\hat{k}}$  for any  $i \leq \hat{k}$ .
- (ii) Suppose that  $\max \left\{ \frac{v_1}{4}, \frac{k_0-1}{\sum_{i \in \mathcal{K}_0} 1/v_i} \right\} = \frac{k_0-1}{\sum_{i \in \mathcal{K}_0} 1/v_i}$ . From Franke et al. (2013), [5], we know that  $X^{*,BLC} \geq \frac{k_0-1}{\sum_{i \in \mathcal{K}_0} 1/v_i}$  because the expression on the right-hand side is not only the contest revenue under the zero head start  $\delta_0 = (0, \dots, 0)$  but also the equilibrium revenue under the unbiased lottery, i.e., where a neutral multiplicative bias  $\alpha^{BLC} = (1, \dots, 1)$  is applied. Note also, that this inequality becomes strict if  $v_1 > v_2$ .

□

Proposition 5.2 has two important implications: Firstly, it shows that the all-pay auction with optimal head starts revenue-dominates any symmetric or asymmetrically biased all-pay auction or lottery contest and any all-pay auction or lottery contest with non-optimal head starts. Hence, a contest organizer interested in maximizing revenue should resort to the all-pay auction framework with optimal head starts. Secondly, the established revenue-ranking implies that head starts are specifically effective for increasing revenue in highly competitive situations like the all-pay auction framework while they are comparatively ineffective in less competitive situations like the lottery contest (in fact, Proposition 4.7 implies that for some degree of heterogeneity using positive head starts even results in lower contest revenue than using zero head starts). Hence, the

difference in maximal revenue using optimal head starts in all-pay auctions versus lottery contests is substantially higher than the respective revenue-difference using biases between the two frameworks. Optimal head starts in the all-pay auction *increase* total revenue beyond the level resulting from optimal biases; while any positive head start in the lottery contest *decrease* revenue below the level resulting from optimal biases as soon as two players are active in equilibrium.

## 6 Conclusion

Granting head starts are a potentially very effective instrument to increase competitive pressure among competing contestants and therefore to induce higher contest revenue. However, the effectiveness of head starts depends on the underlying contest framework in which it is applied. In the case of the highly discriminatory all-pay auction head starts can always be designed such that the level playing field among the two active contestants is totally balanced which also implies that the equilibrium payoff of the strongest contestant is reduced to zero. This payoff-loss due to increased bidding is transformed one-to-one into additional revenue for the contest organizer which is even more effective than under the optimal multiplicative bias. Hence, an all-pay auction with an optimally specified head start does not only revenue-dominate symmetric all-pay auctions without head starts but also any asymmetrically biased all-pay auction or lottery contest.

In the lottery contest framework, however, head starts are rather ineffective in generating additional revenue. Basically, active contestants that receive a head start reduce effort exertion to the same degree as the amount of the received head start; for them, head starts are perfect substitutes for own effort. As the 'effective' effort (effort plus head start) of directly affected contestants remains constant, non-affected contestants will not change their strategic response either. In the aggregate, contest revenue under head starts tends to be lower such that zero head starts would be the preferred option. The only exception is the case where exactly one player remains active who has to compete de-facto against the sum of applied head starts granted to all other non-active players. This type of head start might in some cases revenue-dominate the zero head start case. However, its potential to generate additional revenue is severely limited due to the fact that all other contestants refrain from exerting any effort. In the end the lottery contest with positive head starts is revenue-dominated by the optimally biased lottery contest and therefore also by the all-pay auction with optimal head start or bias.

A natural extension of our analysis would be to consider simultaneous head starts and biases; that is, the bid  $x_i$  of each player is transformed according to the affine transformation  $\alpha_i x_i + \delta_i$  for all  $i \in \mathcal{N}$ . We conjecture that our results would also hold under this generalization: For the

all-pay auction the lower bound derived in Prop. 3.1 presumably cannot be further increased by allowing additional biases. Likewise in the lottery contest it might not be possible to improve on the revenue generated under the optimal bias by allowing additional head starts. Moreover, affine transformations of the considered type can potentially be generalized even further. For the homogeneous  $n$ -player and the heterogeneous two-player case Dasgupta and Nti (1998), [4], and Nti (2004), [12], show that a contest game involving any positive and increasing transformation is strategically equivalent to a contest game with an appropriately specified affine transformation as above. Whether their results can be extended to the heterogeneous  $n$ -player case is left for future research.

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