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**Wirtschaftswissenschaftliche Fakultät
der Eberhard-Karls-Universität Tübingen**

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Risky Corporate Bonds**

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A Note on the Valuation of Risky Corporate Bonds

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Abstract

Simple formulas for the price of corporate discount and coupon bonds are found using the Longstaff and Schwartz [1995] valuation approach for the debt claims of a firm, where default is triggered by a special state variable: the firm's asset-to-debt-ratio. Instead of keeping the total amount of debt constant over time, it is shown that closed form solutions exist under the alternative assumption that the level of leverage is expected to remain constant over time under the risk-neutralized measure. This encourages a more conservative view on the capital structure policy of the firm which might be appropriate in case the firm is neither willing nor able to reduce its expected level of leverage considerably over time.

1 Introduction

In this note a variant of the corporate bond valuation model by Longstaff and Schwartz [1995] is developed. As an extension of the Black and Cox [1976] model the approach of Longstaff and Schwartz [1995] (hereafter LS) not only incorporates interest rate risk explicitly into the analysis but also opens a new research agenda in the sense that the evolution of the firm's capital structure is no longer tied up with the payoffs of any individual (debt-)claim on the firm's assets. This separation allows for modelling the lower threshold for the value of the firm's assets, where default of the firm is assumed to occur, independently from the state-contingent payoffs of the claim under consideration. This has the advantage, that the model can be applied in real world scenarios of complex capital structures including multiple issues of debt and without full knowledge of the priority rules among corporate claimants. In the LS model this lower threshold, which corresponds to the total amount of debt issued by the firm, is kept constant over time.

A different valuation model for risky corporate bonds is presented which preserves most of the advantages of the LS approach. Our point of departure is a different threshold value where default and reorganization of the firm occurs. This barrier is assumed to grow with

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time, reflecting not only time value of money but also the fact that the value of the firm as well as its debt is expected to grow with time, keeping the expected level of leverage constant in a way explained later on. While this new assumption about the evolution of the firm's capital structure is theoretically valuable in its own respect, because it defines a different type of benchmark in case the firm is not able to reduce its debt-ratio over time, in addition we are rewarded with a dramatic simplification of the mathematical structure of the valuation problem: The resulting formula can be evaluated numerically as easily and fast as the celebrated Black and Scholes formula.¹

There are more formulas in the literature which are as simple as ours. Claessens and Pennacchi [1996] present a model for the valuation of Mexican bonds. To characterize the developing country's default risk they use an abstract Gaussian state variable which is assumed to be uncorrelated with the term structure of default-free interest rates. Moreover, they assume that the level of the state variable is independent of the level of interest rates. Compared to that, our corporate bond valuation problem is more complicated and richer in structure. The threshold, where default occurs, is assumed to grow with time. This leads us to a state variable which is not only locally correlated with the interest rate but in addition its level partly depends on the level of this rate.

The remainder of the paper is organized as follows: Section 2 gives an outline of the model. Section 3 presents the new closed form solutions for risky corporate bonds. Section 4 compares our model with the LS model and gives some numerical examples and Section 5 concludes.

2 The Model

In this section we briefly summarize the model. The market value of the firm's assets $V(t)$ is assumed to follow a geometric Brownian motion with constant instantaneous expected rate of return α and constant volatility σ :

$$dV(t) = \alpha V(t) dt + \sigma V(t) dw_1(t). \quad (1)$$

We assume for simplicity that there are no dividends paid to the owners of the firm. The short-term interest rate r follows a mean-reverting Ornstein-Uhlenbeck process with constant parameters κ , μ and η :

$$dr(t) = \kappa(\mu - r(t)) dt + \eta dw_2(t) \quad (2)$$

The system is driven by two correlated Brownian motions $w_1(t)$ and $w_2(t)$ with correlation coefficient ρ :

¹Gagnon, Hurley and Johnson [1997] report that the LS model is numerically rather challenging and time consuming, because it requires a nonstandard numerical approximation of an integral of a first passage density. However, as will be shown in the sequel of this note, this is not a reasonable objection against the usage of a contingent claims framework in general.

$$dw_1(t)dw_2(t) = \rho dt. \quad (3)$$

Given these assumptions, any claim $H(V, r, t; T)$ on the firms assets with maturity T having no interim cash-flows fulfills the following version of the fundamental partial differential equation²

$$\frac{1}{2}\sigma^2V^2H_{VV} + \rho\sigma\eta VH_{Vr} + \frac{1}{2}\eta^2H_{rr} + rVH_V + (\pi - \kappa r)H_r - rH + H_t = 0, \quad (4)$$

subject to a final condition at $t = T$ for all $V > V^*$

$$H(V, r, T; T) = g(V, r) \quad (5)$$

and a boundary condition at the reorganization boundary $V = V^*$ for all $t \leq T$

$$H(V^*, r, t; T) = h(r, t). \quad (6)$$

LS define this reorganization boundary as a constant, namely $V^*(t) = K_{LS}$. In contrast to this, we assume a time-dependent reorganization boundary. Given a fixed amount K of debt outstanding at maturity T of the claim, we assume that default and reorganization of the firm takes place whenever the value of the firm $V(t)$ hits the discounted value of this amount, which is

$$V^*(t) = KD(r, t; T). \quad (7)$$

Hence the total amount of debt outstanding grows in our model with calendar time. This is shown in Figure 1. Whereas the reorganization boundary K_{LS} in the LS model remains constant over time, this new boundary $KD(r, t; T)$ is expected to grow at the riskless rate r .

²See equation (3) in Longstaff and Schwartz [1995].

3 Closed Form Solutions for Risky Corporate Bonds

3.1 Discount Bonds

In this section we give the solution of the above problem for risky corporate discount bonds of the firm. This type of bond receives at maturity T the face value $g(V, r) = 1$ if no default occurs until T . In case a default occurs, we assume, that the claimholder receives a fixed amount $L = 1 - m$ at maturity T , where m is the nominal amount lost by the claimholder. Therefore the risky discount bond can be sold for $h(r, t) = (1 - m)D(r, t; T)$ at any time $t \leq T$ after default.³ As shown in the appendix the solution of this valuation problem can be stated by the following formula

$$P(x, r, t; T) = D(r, t; T) \{1 - m Q(x, \tau)\} \quad (8)$$

where

$$Q(x, \tau) = e^x \mathcal{N}\left(-\frac{x + \frac{1}{2}\tau}{\sqrt{\tau}}\right) + \mathcal{N}\left(-\frac{x - \frac{1}{2}\tau}{\sqrt{\tau}}\right) \quad (9)$$

is the probability under the risk-neutralized measure that a default occurs until T and where the transformed variables x and τ are

$$x(V, t) = \ln V - \ln K + r A(t; T) + B(t; T) \quad (10)$$

$$\tau(t) = \left\{ \sigma^2 + \frac{2}{\kappa} \rho \sigma \eta + \frac{\eta^2}{\kappa^2} \right\} (T - t) - \left\{ \frac{2}{\kappa} \rho \sigma \eta + \frac{\eta^2}{\kappa^2} \right\} A(t; T) - \frac{\eta^2}{2\kappa} A^2(t; T). \quad (11)$$

³The discount factor $D(r, t; T)$ gives the value of an otherwise identical but default-free discount bond. In this case the face value of the bond is guaranteed and therefore independent of $V(t)$. Hence the boundary condition (6) drops out and the PDE (4) is reduced to the Vasicek [1977] model, with solution

$$D(r, t; T) = e^{-A(t; T)r - B(t; T)},$$

where $A(t; T)$ and $B(t; T)$ are the auxiliary functions

$$A(t; T) = \frac{1}{\kappa} (1 - e^{-\kappa(T-t)}),$$

$$B(t; T) = -\frac{1}{\kappa} (1 - e^{-\kappa(T-t)}) R_\infty + (T - t) R_\infty + \frac{\eta^2}{4\kappa^3} (1 - e^{-\kappa(T-t)})^2,$$

and R_∞ is a constant

$$R_\infty = \frac{\pi}{\kappa} - \frac{\eta^2}{2\kappa^3}.$$

This formula can be easily evaluated. No numerical approximation beyond that of the celebrated Black and Scholes formula is needed. As in the LS model the discount bond price depends on the firm's assets-to-debt-ratio only and not on the value of the firm's assets itself. If the state variable, which is the log of this ratio, hits the boundary at $x = 0$ from above for the first time the firm defaults on its debt. However, in contrast to the LS model, where K_{LS} is a constant, in this model the actual amount of debt $KD(r, t; T)$ grows at the same rate r as the value of the firm's assets is expected to grow under the risk-neutralized measure. Given an initially identical debt level $K_{LS} = KD(r, t; T)$ for both models and starting from the same value $V(t)$ of the assets of the firm, the probability of default $Q(x, \tau)$ in our model is always greater than in the LS model. This is due to the fact, that the assumption on the evolution of the capital structure of the firm over time is less "optimistic" in our model than in the LS model.

For later usage we define the yield spread $YS(x, r, t; T)$ for this risky discount bond compared to its default-free counterpart as

$$YS(x, r, t; T) = -\frac{1}{(T-t)} \ln \frac{P(x, r, t; T)}{D(r, t; T)}. \quad (12)$$

3.2 Coupon Bonds

Most important, the value additivity feature of the LS model is preserved here. This means that risky corporate coupon bonds can be valued simply as portfolios of risky corporate discount bonds. This facilitates the valuation procedure considerably. The formula for the price of a risky corporate coupon bond P^c with face value F and discrete coupon payments c being paid annually simply becomes

$$P^c(x, r, t; T) = \sum_{j=0}^{trunc(T)} cP(x, r, t; T-j) + FP(x, r, t; T), \quad (13)$$

where the function value $trunc(T)$ is the largest integer smaller than T . The yield to maturity $y^c(x, r, t; T)$ of the risky corporate coupon bond can be stated as the implicit solution of the equation

$$\sum_{j=0}^{trunc(T)} ce^{-y^c(x, r, t; T) \cdot (T-j)} + Fe^{-y^c(x, r, t; T) \cdot T} = P^c(x, r, t; T). \quad (14)$$

Using formulas (13) and (14) a second time for an otherwise identical but default-free coupon bond $D^c(x, r, t; T)$, the resulting yield spread $YS^c(x, r, t; T)$ for the risky corporate coupon bond is the difference of these two yields.

4 Discussion

4.1 Comparison with the LS Model

We compare our solution with the LS model, using a scenario for extremely risky corporate debt drawn from their paper. With $x = \ln(1.05)$ the value of the firm's assets is assumed to be only slightly above the actual value of the firm's debt. The other parameter values are $m = 0.9$, $r = 0.04$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\pi = 0.06$, $\kappa = 1.00$ and $\eta^2 = 0.001$.

Figure 2 shows the resulting discount-bond prices using these parameter values for both models. $P(T)$ is the risky corporate discount-bond price using formula (8), $P_{LS}(T)$ is the corresponding discount-bond price using the Longstaff/Schwartz [1995] model and $D(T)$ is the price of a standard discount-bond with no default-risk. In this extreme example only 10% of the face value of the bond are recovered by the bondholders in case of default. In principle the qualitative behaviour of both models is quite similar. However, due to the fact that - ceteris paribus - for identical initial values for x the probability of default under the risk-neutralized measure is always greater in our model than in the LS model, the prices $P_{LS}(T)$ are strictly higher than the corresponding prices $P(T)$ for all maturities T . This effect is caused by different assumptions about the capital structure policy of the firm over time. In the LS model we find the implicit assumption that on average the firm is able and willing to reduce the level of leverage considerably, keeping the amount of debt constant over time, while the total value of the firm is expected to increase over time. In our model, we assume that the level of leverage is expected to remain constant over time under the risk neutral measure once risk aversion has been accounted for. Which of these two scenarios lends itself better to a given corporate debt policy is not a theoretical but a practical issue.

4.2 Comparison of Yield Spreads

To demonstrate the potential of the model briefly, we vary the values of x and m , keeping the other parameters unchanged.⁴ To reproduce a AAA-rated bond we choose $x = \ln(3.5)$ and $m = 0.6$. Figure 3 shows the yield spreads in basis points of a risky corporate discount bond $YS(T)$ and a risky corporate coupon bond $YS^c(T)$ paying an annual coupon of $c = 0.08$. We observe that the spreads for the coupon bond are significantly lower than for the discount bond. For maturities between ten to twenty years the spreads range from 38 to 67 basis points for the coupon bond and from 47 to 86 for discount bonds. This corresponds with empirical evidence found in Altman [1989]. For the years 1973 - 1987 he reports an average yield spread on AAA-rated bonds of 47 basis points.

To give another example, we reproduce a somewhat riskier bond, similar to a BBB rating. Now we choose $x = \ln(2)$ and $m = 0.5$. Figure 4 shows the results for these two risky corporate bonds. Here, for maturities below eighteen years, the spreads for the coupon bond are lower than for the discount bond. However for longer maturities this effect is reversed. For maturities between ten to twenty years the spreads range here from 185 to 173 basis points for the coupon bond and from 198 to 170 for discount bond. Again this corresponds

⁴See Altman and Kishore [1996], who report average recovery rates by industry and by seniority. Taking seniority into account they also find that the original rating of a bond issue has virtually no effect on recovery rates. By choosing the model's recovery rate $1 - m$ carefully, these findings can be incorporated into the model.

with Altman's findings, who reports a yield spread on BBB-rated bonds of 177 basis points on average.

5 Conclusion

In this note a variant of the corporate bond valuation model by Longstaff and Schwartz [1995] is developed. New and surprisingly simple formulas for the price of corporate discount and coupon bonds are found using the Longstaff and Schwartz [1995] valuation approach for the debt claims of a firm, where default is triggered by a special state variable: the firm's asset-to-debt-ratio. Instead of keeping the total amount of debt constant over time, it is shown that closed form solutions exist under the alternative assumption that the level of leverage is expected to remain constant over time under the risk-neutralized measure.

Because of its beauty and simplicity, the bond price formulas found in this paper may serve as basic building blocks for the analysis of more complicated corporate debt-claims in the presence of default-risk. This is left for future research.

6 Appendix

In this appendix we give the derivation of our risky discount bond formula (8). For the price of a discount bond $P(V, r, t; T)$ the PDE (4) reads

$$\frac{1}{2}\sigma^2 V^2 P_{VV} + \rho\sigma\eta V P_{Vr} + \frac{1}{2}\eta^2 P_{rr} + rV P_V + (\pi - \kappa r)P_r - rP + P_t = 0 \quad (\text{A1})$$

subject to the final condition at $t = T$ for all $V > V^*$

$$P(V, r, T; T) = 1 \quad (\text{A2})$$

and the boundary condition at the reorganization boundary $V = V^*$ for all $t \leq T$

$$P(V^*, r, t; T) = (1 - m)D(r, t; T). \quad (\text{A3})$$

The difficult part now is to find the transformation

$$v(x, \tau) = K^{-1} e^{-\frac{1}{2}x} e^{\frac{1}{8}\tau} D^{-1}(r, t; T) P(V, r, t; T) \quad (\text{A4})$$

with the new variables x and τ as stated in (10) and (11). Once this is done, the problem is simplified considerably and reduced to one space dimension. We end up with the standard heat transfer equation

$$\frac{1}{2}v_{xx} - v_{\tau} = 0 \quad (\text{A5})$$

subject to the transformed initial and boundary conditions

$$v(x, 0) = K^{-1}e^{-\frac{1}{2}x}, \quad (\text{A6})$$

$$v(0, \tau) = K^{-1}(1 - m)e^{\frac{1}{8}\tau}. \quad (\text{A7})$$

The solution of the heat transfer equation on the semi-infinite domain with state space $0 < x < \infty$ and $\tau > 0$ is well known. It reads

$$\begin{aligned} v(x, \tau) = & \int_0^{\infty} \{G(x - \xi, \tau) - G(x + \xi, \tau)\} v(\xi, 0) d\xi - \\ & - \int_0^{\tau} \frac{\partial G(x, \tau - \varphi)}{\partial x} v(0, \varphi) d\varphi, \end{aligned} \quad (\text{A8})$$

where $G(x, \tau)$ is the standard Green's function for the heat transfer equation on the infinite domain with state space $-\infty < x < +\infty$ and $\tau > 0$

$$G(x - \xi, \tau - \varphi) = \frac{1}{\sqrt{2\pi(\tau - \varphi)}} \exp\left\{-\frac{(x - \xi)^2}{2(\tau - \varphi)}\right\}. \quad (\text{A9})$$

Switching back to the original variables we get

$$\begin{aligned} P(V, r, t; T) = & D(r, t; T) e^{\frac{1}{2}x} e^{-\frac{1}{8}\tau} \times \\ & \times \left[\int_0^{\infty} \frac{1}{\sqrt{2\pi\tau}} \left\{ \exp\left(-\frac{(x - \xi)^2}{2\tau}\right) - \exp\left(-\frac{(x + \xi)^2}{2\tau}\right) \right\} e^{-\frac{1}{2}\xi} d\xi + \right. \\ & \left. + \int_0^{\tau} \frac{x}{\sqrt{2\pi(\tau - \varphi)^3}} \exp\left(-\frac{x^2}{2(\tau - \varphi)}\right) (1 - m) e^{\frac{1}{8}\varphi} d\varphi \right]. \end{aligned} \quad (\text{A10})$$

Finally, standard integral manipulations and some rearranging lead us to formula (8) in the text.

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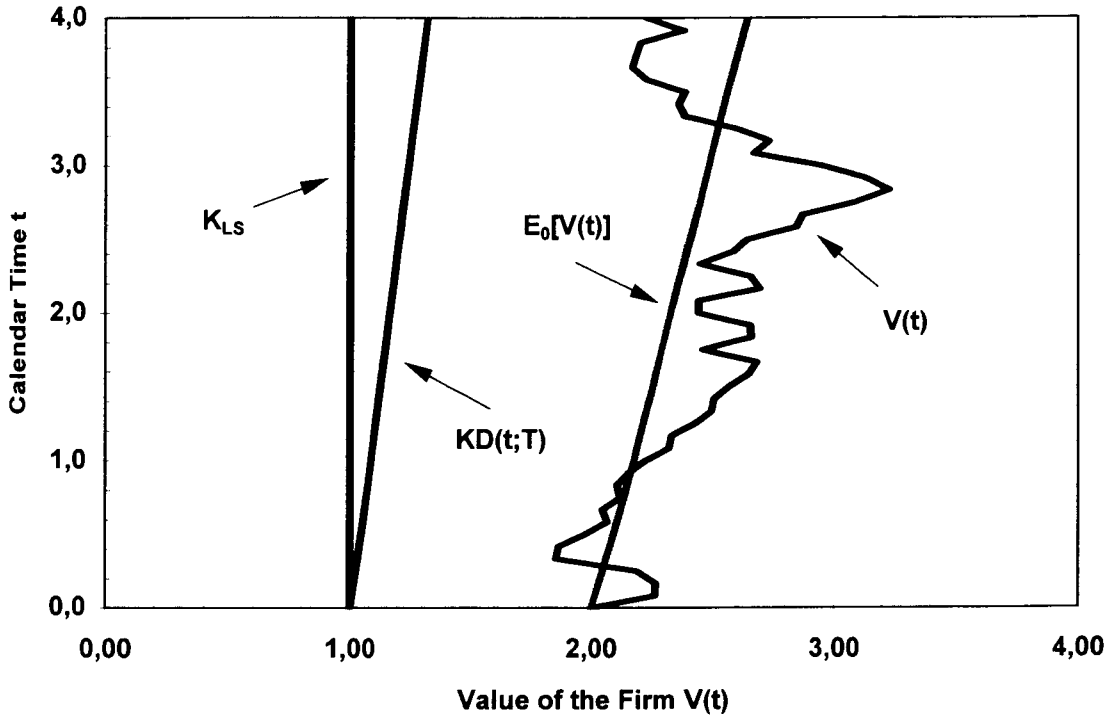


Figure 1: **Illustration of the growing reorganization boundary of the firm.** $V(t)$ is the actual value of the firm, a random variable, and $E_0[V(t)]$ is its expected value with respect to the available information at t_0 . Whereas the reorganization boundary K_{LS} remains constant over time, the new boundary $KD(t;T)$ is expected to grow at the riskless rate r .

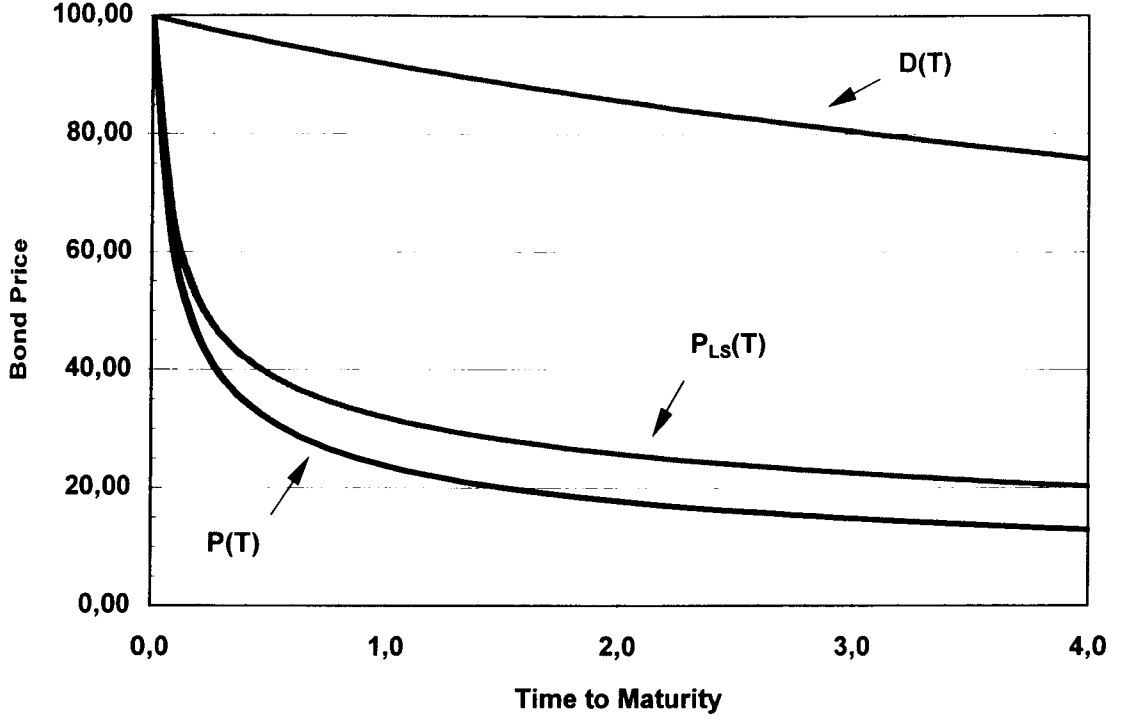


Figure 2: **Comparison of discount bond prices as a function of the maturity of the bond.** $D(T)$ is the price of a discount-bond with no default-risk, $P(T)$ is the risky corporate discount bond price using formula (8) in the paper, and $P_{LS}(T)$ is the corresponding discount bond price using the Longstaff and Schwartz [1995] model. With $x = \ln(1.05)$ the firm's assets-to-debt-ratio is assumed to be only slightly above the actual value of the firm's debt. The other parameter values are $m = 0.9$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\pi = 0.06$, $\kappa = 1.00$ and $\eta^2 = 0.001$.

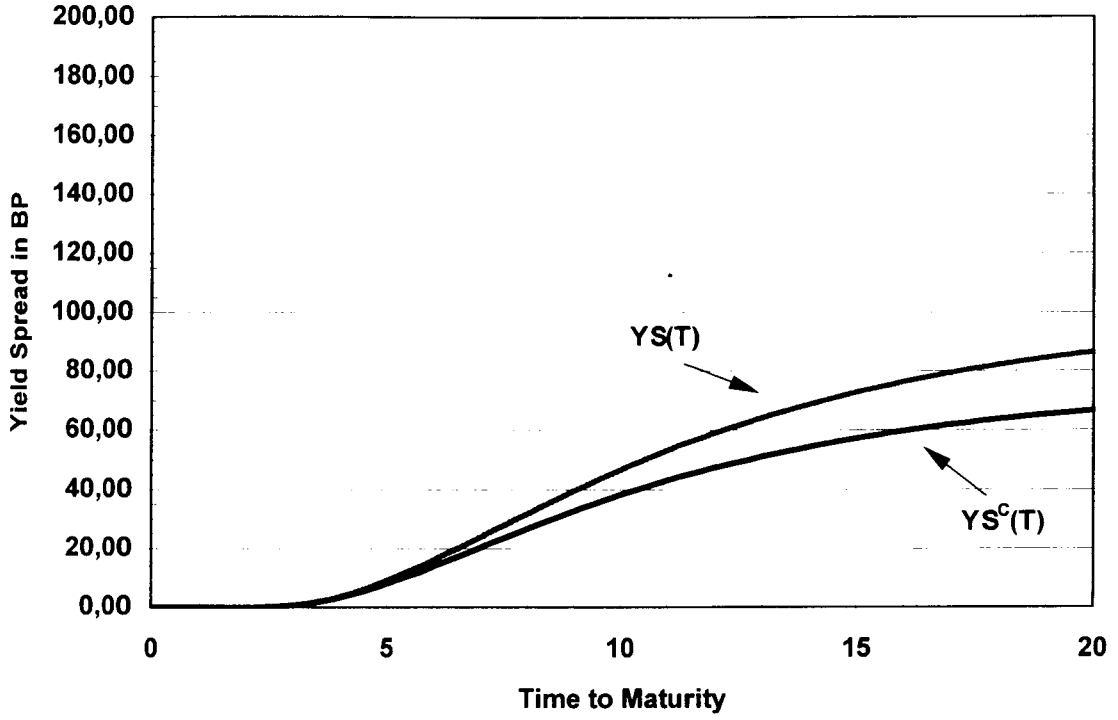


Figure 3: **Model generated yield spreads of risky corporate bonds comparable to a AAA-rating.** $YS(T)$ and $YS^c(T)$ are the spreads of a risky discount and a risky coupon bond with $c = 0.08$ respectively. The parameter values are $x = \ln(3.5)$, $m = 0.6$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\pi = 0.06$, $\kappa = 1.00$ and $\eta^2 = 0.001$.

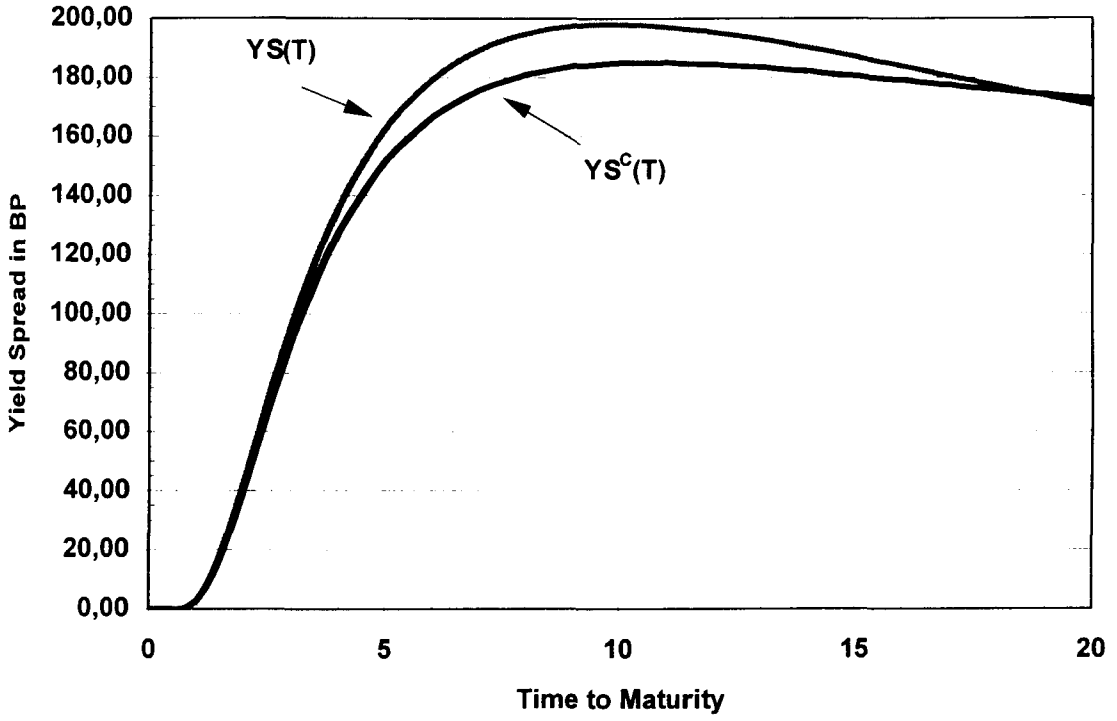


Figure 4: **Model generated yield spreads of corporate bonds comparable to a BBB-rating.** $YS(T)$ and $YS^c(T)$ are the spreads of a risky discount and a risky coupon bond with $c = 0.08$ respectively. The parameter values are $x = \ln(2)$, $m = 0.5$, $\sigma^2 = 0.04$, $\rho = -0.25$, $\pi = 0.06$, $\kappa = 1.00$ and $\eta^2 = 0.001$.